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THE PRINCIPLE THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP

by

Bertram Kostant

Technical Report No. 9

Prepared under Contract AF 49(638)-79

Division File No. 3.22

For

Mathematics Division
Air Force Office of Scientific Research, ARDC,
Washington 25, D. C.

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Department of Mathematics
University of California
Berkeley 4, California

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THE PRINCIPLE THREE-DIMENSIONAL SUBGROUP AND THE BETTI NUMBERS OF A COMPLEX SIMPLE LIE GROUP

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Bertram Kostant

1. Introduction.1. Let $\underline{g}$ be a complex simple Lie algebra and let G be the adjoint group of $\underline{g}$. It is by now classical that the Poincaré polynomial $p_G(t)$ of G factors into the form,

$$p_G(t) = \prod_{i=1}^{\ell} (1 + t^{d_i}) \qquad (1.1.1)$$

where ℓ is the rank of $\underline{g}$ and the d_i are odd integers. In this paper the integers m_i (elsewhere, sometimes $m_i + 1$) defined by $d_i = 2m_i + 1$ will be called the exponents of $\underline{g}$. No doubt one of the reasons the problem of finding the exponents turned out to be as difficult as it was, is that there was no way known by which these numbers could be determined from a direct examination of the structure of $\underline{g}$, particularly the root structure. The first procedure for extracting the exponents from the root structure of $\underline{g}$ was found by R. Bott. The proof of the validity of this procedure depends upon Morse theory.* A second and much simpler way, which we shall presently describe, of "reading off" the exponents from the root structure of $\underline{g}$ was discovered

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* See R. Bott, An application of the Morse theory to the topology of Lie groups, Bull. Soc. Math. France, t. 84 (1956), pp. 251-281.

by Arnold Shapiro. (It is interesting that Shapiro discovered the procedure by misinterpreting the method of Bott.) However, even though one verifies that the numbers produced by this procedure agree with the exponents, unlike the case with Bott's method, the important question of proving that this "agreement" is more than just a coincidence remained open. The procedure is as follows: Let Δ^+ be the set of positive roots relative to some Weyl chamber in a Cartan subalgebra of $\underline{g}$. For each root $\varphi \in \Delta^+$ let $o(\varphi)$, the order of φ, be the sum of coefficients of φ when the latter is expressed as a linear combination of the simple positive roots. Let b_k be the number of roots φ such that $o(\varphi) = k$. Then $b_k - b_{k+1}$ is the number of times k occurs as an exponent of $\underline{g}$. For example, if we apply this to the case where $\underline{g}$ is Lie algebra of the special linear group SL(n, C) then the fact that the number of matrix units e_{ij} such that $j - i = k$, where $1 \leq k \leq n-1$, is greater by one then the number of matrix units e_{ij} such that $j-i = k+1$ accounts for the fact that the exponents of $\underline{g}$ are $1,2,\ldots,n-1$.

After Shapiro informed us of this counting device for the exponents we observed that it could be reformulated as follows: The principle three-dimensional subalgebra $\underline{a}_o$ of $\underline{g}$ is a uniquely defined (up to conjugacy) three dimensional simple subalgebra (TDS) of $\underline{g}$ which can be readily distinguished from other TDS by its properties. [It was discovered almost simultaneously by Dynkin and de Siebenthal (see [6] and [13])

and later used extensively by these authors (see e.g. [7])]. Using standard facts in the representation theory of a TDS it is not difficult to show that the observation of Shapiro is equivalent to the observation that if we decompose the adjoint representation of $\underline{a}_o$ on $\underline{g}$ into a direct sum of irreducible representations then the number of irreducible components is ℓ and the dimensions of these components are d_i, $i = 1, 2, \ldots, \ell$ where the d_i are given by (1.1.1). However, this reformulation of the procedure still does not supply a proof.

A second empirical procedure for finding the exponents was discovered by H. M. Coxeter. He recognized that the exponents can be obtained from a particular transformation γ in the Weyl group, which he had been studying, and which we take the liberty of calling a Coxeter-Killing transformation, in the following manner (see [5]): Let h be the order of γ. Coxeter observed that (1) h satisfies $h\ell = 2r$ where r is the number of positive roots, (2) $m_i \leq h$ for all i and (3) the eigenvalues of γ are ω^{m_i} $i = 1, 2, \ldots, \ell$ where $\omega = e^{\frac{2\pi i}{h}}$. A proof of (2) and (3) would provide, among other things, a proof of a duality in the exponents m_i observed by Chevalley (see [3], p. 24) since non-real eigenvalues of γ necessarily occur in conjugate pairs. Requiring (1) $h\ell = 2r$ as the only empirically observed fact such a proof was recently obtained by A. J. Coleman (see [4]). A proof that $h\ell = 2r$ will be given in this paper. A second question posed in [4] of showing

that $h = 1 + o(\psi)$ where ψ is the highest root will also be settled here.

It will be the main result of this paper to establish a direct relationship between the principle TDS, its adjoint representation of $\underline{g}$, and the transformation of Coxeter-Killing. The proof that the procedure of Shapiro yields the exponents is then a direct consequence.of this relationship. A major role here is played by a particular conjugate class in G, the elements of which we call principle elements of G. It is shown than an element of G which induces a transformation of Coxeter-Killing on a Cartan subalgebra is necessarily a principle element. On the other hand a principle element of G belongs to the subgroup corresponding to a principle TDS and is sufficiently specialized in that subgroup so that the adjoint representation of a principle TDS on $\underline{g}$ is determined by its eigenvalues. That the conjugate class of principle elements is truly a distinguished one in G may be judged from the following characterization (one of several) of principle elements. Let A be an arbitrary regular element of G. Let k be the order of A (possibly ∞). Then $k \geq h$ and $k = h$ if and only A is principle.

An important role is also played here by two distinguished classes of $\underline{g}$, the elements of which we have, respectively, called principle nilpotent and cyclic (the latter name is derived from the transformation properties of such elements in

the case when $\underline{g}$ is the Lie algebra of $SL(n,C)$). As is the case with the principle elements of G these elements can be given simple characterizations (Corollary 5.3, Theorem 9.2 and Corollary 9.3).

In this paper §§ 1-4 are devoted to the theory of the general TDS. The main results here are Theorems 3.6 and 4.2. With regard to questions of conjugacy these theorems, respectively, strengthen and extend well known theorems of Jacobson-Morosov and Malcev. It is shown here also that the conjugate classes of TDS in $\underline{g}$ are in a natural one-one correspondence with the conjugate classes of nilpotent elements in $\underline{g}$.

In §5 the theory of the principle TDS is taken up. For the most part the theorems here are devoted to characterizing principle TDS among all TDS and principle nilpotent elements among all nilpotent elements. The result here which is used most often in the remainder of the paper is Corollary 5.3.

The main results of the paper are given in §§ 6-9.

2. Preliminaries and the complex three dimensional simple Lie algebra. 1. Let $\underline{g}$ be a complex semi-simple Lie algebra. Let n and ℓ be respectively the dimension and rank of $\underline{g}$. As usual the linear transformation

$$y \longrightarrow [x,y]$$

is designated by adx and $x \longrightarrow \mathrm{ad}\, x$ is the adjoint representation of $\underline{g}$ on itself. If $\underline{u}$ is a Lie subalgebra of $\underline{g}$ the

mapping $x \longrightarrow \operatorname{ad} x$ for $x \in \underline{u}$ will be called the adjoint representation of $\underline{u}$ on $\underline{g}$. We distinguish two types of elements in $\underline{g}$. An element $x \in \underline{g}$ is called nilpotent if $\operatorname{ad} x$ is nilpotent and is called semi-simple if $\operatorname{ad} x$ is completely reducible, that is, (since we are dealing with the complex numbers) if $\operatorname{ad} x$ is diagonalizable.[1] Only the zero element is both nilpotent and semi-simple.

We recall that any element x contained in a Cartan subalgebra $h \subseteq \underline{g}$ is necessarily semi-simple and conversely every semi-simple element may be embedded in a Cartan subalgebra. (See e.g. [8], p. 119).

For any element $x \in \underline{g}$ let $\underline{g}^x$ designate the kernel of $\operatorname{ad} x$. We recall that x is called regular when x is semi-simple and $\underline{g}^x$ is a Cartan subalgebra.

2.2. Let G designate the adjoint group of $\underline{g}$. Since this is the only group associated with $\underline{g}$ that we shall consider the operation of exponentiation (Exp) will be always understood to go from $\underline{g}$ to G.

Elements x and y in $\underline{g}$ are called conjugate if there exists $A \in G$ such that $Ax = y$.

2.3. The simplest complex semi-simple Lie algebra (up to isomorphism) is $\underline{a}_1$, the Lie algebra of all complex 2×2 matrices of trace zero. In this case $n = 3$ and $\ell = 1$. One

---

1 It is not difficult to show that this very same definition [of semi-simple and nilpotent elements] may be achieved using any faithful representation of $\underline{g}$ in place of the adjoint representation. This follows, e.g. from Lemma 5.4 and well known facts in the representation theory of $\underline{g}$.

knows that any three dimensional complex semi-simple Lie algebra is isomorphic to $\underline{a}_1$.

By conjugating any element of $\underline{a}_1$ into Jordan canonical form the following is apparent:

(a) Every non-zero element in $\underline{a}_1$ is either semi-simple or nilpotent,

(b) the set of all non-zero nilpotent elements in $\underline{a}_1$ form a single conjugate class,

(c) if $x,y \in \underline{a}_1$ are semi-simple, $x \neq 0$, then y is conjugate to a unique, up to sign, scalar multiple of x. (Note also that the set of non-zero semi-simple elements coincides with the set of regular elements in $\underline{a}_1$).

2.4. Let $\underline{a}$ be a 3-dimensional complex simple Lie algebra. We recall further facts in the structure theory of $\underline{a}$. Let $x \in \underline{a}$ be a regular element. Then the eigenvalues of ad x are a, 0 - a for some non-zero complex number a. We may modify x by scalar multiplication so that a and -a take the values 1 and -1. This defines x uniquely up to conjugacy. We thus isolate a particular conjugate class in $\underline{a}$. The elements in this class will be called mono-semisimple.

Let $x \in \underline{a}$ be a mono-semisimple element. Let $e_+ \in \underline{a}$ be a non-zero eigenvector of adx for the eigenvalue 1; e_+ is unique up to a non-zero scalar. Then e_-, an eigenvector of adx for the value -1, is uniquely determined by condition $[e_+, e_-] = x$. One thus has the commutation relations

$$(1)\quad [x, e_+] = e_+$$
$$(2)\quad [x, e_-] = -e_- \qquad (2.4.1)$$
$$(3)\quad [e_+, e_-] = x$$

for the bases x, e_+, e_- of $\underline{a}$.

2.5. When $\underline{a} = \underline{a}_1$ the elements x, e_+, and e_- may be realized by the matrices

$$x = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad e_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

For each positive integer d there exists up to equivalence one and only one linear irreducible representation of $\underline{a}$ having that dimension (For a complete treatment of the representation theory of $\underline{a}_1$ see [12, Exposé no. 10]. To describe an irreducible representation π_d of $\underline{g}$ on an d-dimensional vector space V first define the number k by the relation $d = 2k + 1$. It is of course to be noted that k is an integer if and only if d is odd. We may then find a basis v_j, $j = k, k-1, \ldots, -k$ of V, where the vectors are each unique up to a scalar factor, satisfying the following condition

$$\pi_d(x)v_j = j \cdot v_j \quad .$$

The behavior of $\pi_d(e_+)$ and $\pi_d(e_-)$ on the one dimensional spaces (v_j) is given by

$$\pi_d(e_+)(v_j) = (v_{j+1})$$
$$\pi_d(e_-)(v_j) = (v_{j-1})$$

where $v_{k+1} = v_{-k-1} = 0$. Thus v_j is an eigenvector for $\pi_d(e_+)\pi_d(e_-)$ and $\pi_d(e_-)\pi_d(e_+)$ where in fact

$$\pi_d(e_+)\pi_d(e_-)v_j = \tfrac{1}{2}(k-j+1)(k+j)v_j$$

and

$$\pi_d(e_-)\pi_d(e_+)v_j = \tfrac{1}{2}(k-j)(k+j+1)v_j$$

It follows that with respect to the ordering $v_k, v_{k-1},\ldots,v_{-k}$ of the basal elements, when the latter are suitably modified by scalar multiplication, one obtains for $\pi_d(x)$, $\pi_d(e_+)$ and $\pi_d(e_-)$ the matrices

$$\pi_d(x) = \begin{vmatrix} k & & & \\ & k-1 & & 0 \\ & & \ddots & \\ 0 & & & -k \end{vmatrix}$$

$$\pi_d(e_+) = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1(2k) & & 0 \\ & 0 & 2(2k-1) & \\ & & 0 \ddots 0 & 2k(1) \\ 0 & & & 0 \end{vmatrix}$$

$$\pi_d(e_-) = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & & & \\ \sqrt{1(2k)}0 & & & \\ & \sqrt{2(2k-1)}0 & \ddots & \\ & 0 & & 0 \\ & & & \sqrt{2k(1)}0 \end{vmatrix}$$

Several facts are to be noted. Among those we shall require are

(a) The dimension d of V is odd or even according as the eigenvalues of $\pi_d(x)$ are all integers or all half-integers.[2]

(b) The eigenvalues of $\pi_d(x)$ all occur with multiplicity 1 and the real number j is an eigenvalue of $\pi_d(x)$ if and only if $d - (2|j|+1)$ is a non-negative even integer.

(c) The number k where $2k+1 = d$ may be characterized as the highest eigenvalue of $\pi_d(x)$. Furthermore the one-dimensional eigenspace for the eigenvalue may be characterized as the kernel of $\pi_d(e_+)$.

Now assume π is an arbitrary, not necessarily irreducible, representation of $\underline{a}$ on the finite-dimensional vector space V. One knows π may be decomposed into a direct sum of irreducible representations. It follows then that one knows π up to equivalence as soon as the dimensions of the irreducible components of π are given. That is, if n_k denotes the number of such components having dimension $2k+1$ then π is given when the sequence n_k, $k = 0, \frac{1}{2}, 1, \ldots$ is known.

(d) The problem of finding the sequence n_k can be reduced to an investigation of the kernel $W \subseteq V$ of $\pi(e_+)$. In fact it follows from (c) that

$$\dim W = n_0 + n_{\frac{1}{2}} + n_1 + \ldots .$$

Furthermore W is stable under $\pi(x)$ and if $w \in W$ is any eigenvector of $\pi(x)$, w may be embedded in an irreducible component of π. Hence if $k_1, k_2, \ldots, k_p$ are the eigenvalues of $\pi(x)$ on

---

2. We use the word half-integer to designate all numbers of the form $m + \frac{1}{2}$ where m is an integer.

W the dimensions of the irreducible components of π are respectively $2k_1+1$, $2k_2+1$, ..., $2k_p+1$. (Note that the k_1 are non-negative.)

(e) The space V admits a canonical direct sum decomposition

$$V = V^E + V^O$$

where V^E is spanned by eigenvectors of $\pi(x)$ belonging to half-integral eigenvalues and V^O is spanned by eigenvectors of $\pi(x)$ belonging to integral eigenvalues. It follows immediately from (a) that V^E and V^O are both stable subspaces for the representation π and that in the complete reduction of $\pi|V^E$ only irreducible representations of even dimension appear and the complete reduction of $\pi|V^O$ yields only irreducible representations of odd dimension.[3]

(f) Now let V_j be the eigenspace of $\pi(x)$ for the eigenvalue j. Clearly $\dim V_j = \dim V_{-j}$. Furthermore if j is non-negative it follows from (b) that

$$\dim V_j = n_j + n_{j+1} + n_{j+2} + \cdots \quad .$$

The statement (f) has the following 2 consequences:

(g) The dimension of V_0, that is the dimension of the kernel of $\pi(x)$ (nullity of $\pi(x)$) equals $n_0 + n_1 + n_2 + \cdots$.

(h) If j is non-negative

$$\dim V_j - \dim V_{j+1} = n_j \quad .$$

---

3. If π is a representation on a vector space V and $U \subseteq V$ is stable under π then $\pi|U$ denotes the representation on U obtained by restricting π to U.

Finally we shall require

(i) If k is the maximal eigenvalue of $\pi(x)$

$$V = \sum_{p=-2k}^{2k} V_{\frac{p}{2}}$$

is a direct sum and $\sum_{p=1}^{2k} V_{\frac{p}{2}}$ lies in the range of $\pi(e_+)$.

3. Nilpotent elements and TDS. 1. Let $\underline{g}$ be a complex semi-simple Lie algebra. Consider the question of determining all three dimensional simple subalgebras (TDS) in $\underline{g}$. If $\underline{a} \subseteq \underline{g}$ is a TDS then by considering the adjoint representation of $\underline{a}$ on $\underline{g}$ it follows from the representation theory of $\underline{a}$ outlined in § 2.5 that any nilpotent element of $\underline{a}$ is necessarily nilpotent in $\underline{g}$ and any semi-simple element of $\underline{a}$ is a semi-simple element of $\underline{g}$ (Also see footnote 1). Since $\underline{a}$ has only semi-simple and nilpotent elements (see § 2.3(a)) the question arises (1) which nilpotent and semi-simple elements of $\underline{g}$ can be embedded in a TDS and (2) how does one find all such subalgebras. We shall first consider the case of nilpotent elements.

A theorem of Morosov asserts that every nilpotent element of $\underline{g}$ can be embedded in a TDS. (See [11].) However, his proof was incomplete. Later in [9] Jacobson gave a correct proof of this result. Since the proof leads into Theorem 3.6 we shall give it here. The proof requires Lemma 3.3. With the exception of the proof of Lemma 3.3 the proof is the same as the one given by Jacobson.

3.2. A famous result of Jacobson asserts that if A and B are linear transformations on a finite dimensional space V with the condition that [A,B] commutes with A then [A,B] is nilpotent. If in addition A is assumed nilpotent the following lemma (which is no doubt known) asserts that AB is nilpotent.

Lemma 3.2. Let A and B be linear transformations on a finite dimensional space V. Assume A is nilpotent and

$$[A,[A,B]] = 0$$

Then AB is nilpotent.

Proof. Let V^k be the kernel of A^k We wish to show AB leaves V^k invariant. The result is obvious if $k = 0$. Assume the result is known to be true for $k = r$. Let $x \in V^{r+1}$

$$ABx = [A,B]x + BAx \quad .$$

Apply A^{r+1} to both sides. Then

$$A^{r+1}(AB)x = [A,B]A^{r+1}x + A^{r+1}BAx$$

$$= 0 + A^r(AB)Ax \quad .$$

But $Ax \in V^r$ and by our assumption $(AB)Ax \in V^r$. Thus $A^r(AB)Ax=0$. Hence $ABx \in V^{r+1}$.

Assume AB is not nilpotent. Then there exists a scalar λ, $\lambda \neq 0$ and a vector $x \in V$, $x \neq 0$ such that

$$ABx = \lambda x \quad . \tag{3.2.1}$$

Let k be the smallest integer such that $x \in V^{k+1}$.

Now

$$A^kABx = A^k[A,B]x + A^kBAx \quad .$$

But $A^kB = BA^k + k[A,B]A^{k-1}$. Thus

$$A^kBAx = BA^{k+1}x + k[A,B]A^kx$$

$$= 0 + k[A,B]A^kx \quad .$$

Hence

$$A^kABx = (k+1)[A,B]A^kx \quad .$$

But by (3.2.1)

$$A^kABx = \lambda A^kx \quad .$$

Hence $[A,B]A^kx = \frac{\lambda}{k+1} A^kx$.

Since $A^kx \neq 0$ this contradicts Jacobson's lemma asserting the nilpotence of $[A,B]$. Q.E.D.

3.3. Now for any $x,y \in \underline{g}$ let

$$(x,y) = \text{tr ad } x \text{ ad } y$$

be the Cartan-Killing bilinear form B on $\underline{g}$. Using the non-singularity of B on $\underline{g}$ we can now prove the following lemma. Lemma 3.3 is crucial in the proof that any nilpotent element $e \in \underline{g}$ can be embedded in a TDS of $\underline{g}$.

It is clear that if e is contained in a TDS then e must lie in the range of $(\text{ad } e)^2$ since according to § 2.3(b), e can play the role of e_+ in the commutation relations (2.4.1). In particular it is interesting enough to observe then that for any non-zero nilpotent element e, $(\text{ad } e)^2 \neq 0$. (The degree of nilpotency of ad e is greater than 2.)

Lemma 3.3. *Let* $e \in \underline{g}$ *be a nilpotent element. Then* e *is in the range of* $(\text{ad } e)^2$.

Proof. The invariance of B under the adjoint representation implies that, for any $z \in \underline{g}$, ad z is skew-symmetric with respect to B and hence $(\text{ad } z)^2$ is symmetric. In particular this is true for $z = e$. But now if A is a symmetric operator (with respect to B) on $\underline{g}$ and if R_A and K_A are, respectively, the range and kernel of A then

$$(R_A, K_A) = 0 \quad .$$

By the non-singularity of B, then, to show an element z lies in R_A, it suffices to show $(z,y) = 0$ for all $y \in K_A$. Letting $A = (\text{ad } e)^2$ to prove the lemma it suffices to show that

$$[e, [e,y]] = 0 \qquad (3.3.1)$$

implies

$$(e,y) = 0 \quad .$$

But under the adjoint representation (3.3.1) becomes

$$[\text{ad } e, [\text{ad } e, \text{ad } y]] = 0$$

Since ad e is nilpotent we apply Lemma 3.2 to assert that ad e ad y is nilpotent. But then by definition of B it is clear that $(e,y) = 0$. Q.E.D.

3.4. By Lemma 3.3. e may be written as

$$[[f,e],e] = e \qquad (3.4.1)$$

for some $f \in \underline{g}$. Let $x = [f,e]$ so that $[x,e] = e$.

Lemma 3.4. Let $\underline{g}^e$ be the kernel of ade. Then $\underline{g}^e$ is invariant under ad x. Furthermore if m is the smallest integer such that $(\text{ad } e)^{m+1} = 0$ then

$$\prod_{p=0}^{m} (\text{ad } x - \tfrac{p}{2})$$

vanishes on $\underline{g}^e$.

Proof. The space $(\text{ad } e)^p \underline{g}$ is the range of $(\text{ad } e)^p$. We define a sequence of subspaces $\underline{d}_p$ of $\underline{g}^e$, $p = 0,1,\ldots,m+1$ where

$$\underline{g}^e = \underline{d}_0 \subseteq \underline{d}_1 \subseteq \cdots \subseteq \underline{d}_{m+1} = 0 \qquad (3.4.2)$$

by letting

$$\underline{d}_p = (\text{ad } e)^p \underline{g} \cap \underline{g}^e \quad .$$

First observe that $\underline{g}^e$ is invariant under ad x. Indeed if $y \in \underline{g}^e$

$$\begin{aligned} [e,[x,y]] &= [[e,x],y] \\ &= -[e,y] \\ &= 0 \quad . \end{aligned}$$

We now show that

$$(\text{ad } x - \tfrac{p}{2}) : \underline{d}_p \longrightarrow \underline{d}_{p+1} \quad . \qquad (3.4.3)$$

Let $y \in \underline{d}_p$. Then $y = (\text{ad } e)^p z$ for some $z \in \underline{g}$. Now since $[x,e] = e$ clearly

$$[\text{ad } x, (\text{ad } e)^p] = p(\text{ad } e)^p \quad .$$

Thus

$$[\text{ad } x, (\text{ad } e)^p]z = py,$$

or

$$\begin{aligned} [x,y] - p\cdot y &= (\text{ad } e)^p [x,z] \\ &= (\text{ad } e)^p [[f,e]z] \qquad (3.4.4) \\ &= (\text{ad } e)^{p+1} [z,f] + (\text{ad } e)^p \text{ad } f[e,z]. \end{aligned}$$

But
$$[(\text{ad } e)^p, \text{ad } f] = -\sum_{i=0}^{p-1} (\text{ad } e)^i \text{ad } x(\text{ad } e)^{p-1-i}$$
$$= \frac{p(p-1)}{2} (\text{ad } e)^{p-1} - p \text{ ad } x(\text{ad } e)^{p-1} .$$

Applying this to $[e,z]$ we obtain

$$(\text{ad } e)^p \text{ad } f[e,z] = \text{ad } f(\text{ad } e)^p[e,z] +$$
$$+ \frac{p(p-1)}{2}(\text{ad } e)^p z - p[x,(\text{ad } e)^p z]$$
$$= [f,[e,y]] + \frac{p(p-1)}{2} y - p[x,y]$$
$$= \frac{p(p-1)}{2} y - p[x,y] \quad .$$

But then (3.3.4) becomes

$$(p+1)[x,y] - \frac{p(p+1)}{2} y = (\text{ad } e)^{p+1}[z,f]$$

or

$$[x,y] - \frac{p}{2} y \in \underline{d}_{p+1} \quad .$$

This proves (3.4.3). It then follows immediately from (3.4.2) that

$$\prod_{p=0}^{m} (\text{ad } x - \frac{p}{2})$$

vanishes on $\underline{g}^e$. Q.E.D.

It is an immediate consequence of Lemma 3.4 that ad x is completely reducible on $\underline{g}^e$ and that its eigenvalues are restricted to non-negative integers and half-integers. In particular what is essential for us at this point is

Corollary 3.4. <u>The linear transformation</u> ad x + 1 <u>is non-singular on</u> $\underline{g}^e$.

We can now prove

Theorem 3.4 (Jacobson-Morosov). Every nilpotent element of a complex semi-simple Lie algebra can be embedded in a TDS.

Proof. Let e be nilpotent, $e \neq 0$, and let f and x be defined as in §3.4.[4]

In case $[x,f] = -f$ we would be done. That is, x, e, f would satisfy the desired commutation relations. The problem is to modify f so that this relation is satisfied without destroying the relation (3.4.1). Even if $[x,f] + f \neq 0$ we still have

$$[[x,f] + f, e] = 0$$

as one easily checks Thus $[x,f] + f \in \underline{g}^e$. But now by Corollary 3.4 since ad x + 1 is non-singular on $\underline{g}^e$ there exists a unique $g \in \underline{g}^e$ such that

$$[x,f] + f = [x,g] + g \quad .$$

Then writing e_+ for e and letting $e_- = f-g$ it follows that

$$\begin{aligned} [x,e_+] &= e_+ \\ [x,e_-] &= -e_- \qquad (3.4.5) \\ [e_+,e_-] &= x \quad . \end{aligned}$$

This proves Theorem 3.4 as soon as one notes that x, e_+ and e_- must be linearly independent. Q.E.D.

3.5. Any set of non-zero elements x, e_+ and e_- in $\underline{g}$ satisfying the commutation relations (3.4.5) will henceforth be called an S-triple (to be written $\{x,e_+,e_-\}$). The element x will

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4. The theorem holds when $e = 0$ once we know the existence of a single TDS. The existence of a TDS follows from the proof since $\underline{g}$ contains non-zero nilpotent elements.

be called the neutral element of the S-triple and e_+ (resp. e_-) will be called the nil-positive (resp. nil-negative) element of the S-triple. It is obvious that the elements of an S-triple form a basis of a TDS. Two S-triples are called conjugate if there exists $A \in G$ which carries one set onto the other.

Given a non-zero nilpotent element $e \in \underline{g}$ we wish now to find all S-triples which contain e as the nil-positive element. (By § 2.3(b) this yields all TDS which contain e).

First we note the following corollary of the proof of Theorem 3.4.

Corollary 3.5. Let $e \in \underline{g}$ be nilpotent, $e \neq 0$, then x and e are respectively the neutral and nil-positive elements of an S-triple if and only if (1) x is in the range of ad e and (2) $[x,e] = e$. Furthermore if x and e satisfy these conditions such an S-triple system is unique (and hence x and e are contained in just one TDS).

Proof. The first part of Corollary 3.5 follows from the proof of Theorem 3.4 and the definition of x used in the proof (see (3.4.1)). To prove the uniqueness of the nil-negative element assume that $\{x,e,f_1\}$ and $\{x,e,f_2\}$ are two S-triples. It is obvious then that

$$[e,f_1-f_2] = 0$$

and hence $f_1-f_2 \in \underline{g}^e$. But

$$[x,f_1-f_2] + f_1-f_2 = 0$$

so that f_1-f_2 is an eigenvector of ad x+ 1 on $\underline{g}^e$. By this implies $f_1 = f_2$. Q.E.D.

3.6. As a consequence of Corollary 3.5 the problem of finding all S-triple systems containing e as the nil-positive element reduces to finding all elements $x \in \underline{g}$ which satisfy the conditions of Corollary 3.5. Towards this end define, for $e \in \underline{g}$, the subspace

$$\underline{g}_e = \text{ad } e(\underline{g}) \cap \underline{g}^e \ ,$$

the intersection of the range and kernel of ad e. We now observe that $\underline{g}_e$ is a Lie subalgebra of $\underline{g}$. Indeed $\underline{g}^e$ is a Lie subalgebra of $\underline{g}$. Therefore it suffices only to show that $[u,v] \in \text{ad } e(\underline{g})$ if $u,v \in \underline{g}_e$. Writing $v = [e,w]$ we have, since $u \in \underline{g}^e$,

$$\begin{aligned} [u,v] &= [u,[e,w]] \\ &= [e,[u,w]] \ . \end{aligned}$$

This proves $[u,v] \in \underline{g}_e$. Now let G_e be the subgroup of G corresponding to the subalgebra $\underline{g}_e$. Note that among other things the elements of G_e leave e fixed. We can now state

Theorem 3.6. *Let $e \in \underline{g}$, $e \neq 0$, be nilpotent. Let $\underline{g}_e$ and G_e be as above. Then the elements of $\underline{g}_e$ are all nilpotent (and hence the elements in G_e are unipotent). Let $x \in \underline{g}$ be such that x and e are, respectively, the neutral and nil-positive elements of an S-triple. Then the linear coset $x + \underline{g}_e$ of $\underline{g}_e$ is the set of all neutral elements taken from all S-triples containing e as nil-positive elements. Furthermore any two elements in $x + \underline{g}_e$ are conjugate. Moreover the conjugation can be performed by an element in G_e so that e is fixed under the conjugation. In fact for any $A \in G_e$, $Ax \in x + \underline{g}_e$ and the map*

$$G_e \longrightarrow x + \underline{g}_e$$

defined by making A, $A \in G_e$, correspond to Ax is one-one and onto.

In other words (recalling Corollary 3.5) if {x,e,f} is the S-triple containing x and e the map

$$A \longrightarrow \{Ax,\ e,\ Af\}$$

sets up a one-one correspondence of the group G_e onto the set of all S-triples containing e as nil-positive element . Furthermore Ax ranges over $x + \underline{g}_e$.

Proof. Assume {x,e,f} and {y,e,g} are S-triples and e nil-positive in both cases while x and y are neutral.

Since

$$[x,e] = [y,e] = e$$

it follows that $y-x \in \underline{g}^e$. But clearly $[e,g-f] = y-x$. That is, $y-x \in \text{ad } e(\underline{g})$. Hence $y-x \in \underline{g}_e$ or $y \in x+\underline{g}_e$. Conversely, if $y \in x+\underline{g}_e$ then clearly $[y,e] = e$ and $y \in \text{ad } e(\underline{g})$. Thus applying Corollary 3.5 y is a neutral element of an S-triple containing e.

Now according to Lemma 3.4 $\underline{g}^e$ admits the direct sum decomposition

$$\underline{g}^e = \sum_{p=0}^{m} \underline{b}_{\frac{p}{2}}$$

where m is the smallest integer such that $(\text{ad } e)^{m+1} = 0$ and $\underline{b}_{\frac{p}{2}}$ is the eigenspace of adx in $\underline{g}^e$ belonging to the eigenvalue $\frac{p}{2}$. It is of course clear that

$$\left[\underline{b}_{\frac{p}{2}},\ \underline{b}_{\frac{p'}{2}}\right] \subseteq \underline{b}_{\frac{p+p'}{2}} \qquad (3.6.1)$$

Let $\underline{a}$ be the TDS spanned by x, e and f. Now if π_d is an irreducible representation of $\underline{a}$ on a vector space of dimension d it is clear from § 2.5(i) that the eigenspace belonging to the highest eigenvalue $(\frac{d-1}{2})$, of $\pi_d(x)$ lies in the range of $\pi_d(e)$ if and only if $d \geq 2$. Now decompose $\underline{g}$ into irredicuble subspaces under the adjoint representation of $\underline{a}$ on $\underline{g}$ and apply this fact to the irreducible components. Recalling 2.5(d) it becomes clear then that the subspace $\underline{g}_e$ of $\underline{g}^e$ can be written

$$\underline{g}_e = \sum_{p=1}^{m} \underline{b}_{\frac{p}{2}} \tag{3.6.2}$$

and hence, in particular adx is non-singular on $\underline{g}_e$.

One immediate observation from (3.6.1) and (3.6.2) is that $\underline{g}_e$ is a nilpotent Lie subalgebra of $\underline{g}$. Furthermore, since the eigenvalues of ad x on $\underline{g}_e$ are strictly positive it is clear from a relation similar to (3.6 1) that ad w is nilpotent for every $w \in \underline{g}_e$.[5] That is, the elements of ad $\underline{g}_e$ may be simultaneously trianglized with zeros appearing along the diagonal for every element. It follows then from well known facts concerning linear nilpotent Lie algebras that in such a case G_e is closed, simply connected - that is, G_e is homeomorphic to Euclidean space - and the exponential map

$$\text{Exp} : \underline{g}_e \longrightarrow G_e$$

is one-one and onto. That is, every $A \in G_e$ may be uniquely qritten $A = \text{Exp } w$ for a unique $w \in \underline{g}_e$. But then

$$Ax = x + [w,x] + \frac{[w[w,x]]}{2!} + \ldots \tag{3.6.3}$$

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5. That is, if $\underline{g}_j$ is the eigenspace of ad x on $\underline{g}$ for the eigenvalue j then clearly $[\underline{g}_j, \underline{b}_i] \subseteq \underline{g}_{i+j}$.

Since $\underline{g}_e$ is stable under ad x and since $\underline{g}_e$ is a Lie subalgebra all terms starting from the second on the right side of (3.6.3) lie in $\underline{g}_e$ so that $Ax \in x + \underline{g}_e$.

Now let $v \in \underline{g}_e$. Assume that there exists a unique element $w_j \in \underline{g}_e$ such that (1)

$$w_j \in \sum_{p=1}^{j} \underline{b}_{\frac{p}{2}}$$

and (2)

$$\text{Exp } w_j(x) - (x+v) \in \sum_{p=j}^{m} \underline{b}_{\frac{p}{2}}$$

Now let z_{j+1} be the component of Exp $w_j(x) - (x+v)$ in $\underline{b}_{\frac{j+1}{2}}$. Then if

$$w_{j+1} = w_j + \frac{2}{j+1} z_{j+1}$$

it is clear that $[w_{j+1},x] = [w_j,x] - z_{j+1}$. On the other hand for $i > 1$ it follows from (3.6.1) that the components of $(\text{ad } w_{j+1})^i x$ and $(\text{ad } w_j)^i x$ in $\underline{b}_{\frac{s}{2}}$ are the same for all $s \leq j+1$. Thus

$$w_{j+1} \in \sum_{s=1}^{j+1} \underline{b}_{\frac{s}{2}}$$

and

$$\text{Exp } w_{j+1}(x) - (x+v) \in \sum_{s=j+2}^{m} \underline{b}_{\frac{s}{2}}$$

and furthermore that in satisfying these conditions w_{j+1} is unique. If we define $w_1 = -2v_{\frac{1}{2}}$ where $v_{\frac{1}{2}}$ is the component of v in $\underline{b}_{\frac{1}{2}}$ then w_1 uniquely satisfies (1) and (2) when $j = 1$.

Thus we have proved inductively that there exists a unique $w \in \underline{g}_e$ such that

$$\text{Exp } w\,(x) = x+v. \qquad \text{Q.E.D.}$$

Note. It is useful to observe that the proof, above, of the statement that Ax ranges over $x + \underline{g}_e$ when A ranges over G_e depends essentially on just two facts, (1), the nilpotence of $\underline{g}_e$ and (2), the non-singularity of ad x on $\underline{g}_e$.

We can now supplement Theorem 3.4 with

Corollary 3.6. Let $e \in \underline{g}$ be nilpotent, $e \neq 0$, and assume $e \in \underline{a}_1 \cap \underline{a}_2$ where $\underline{a}_1$ and $\underline{a}_2$ are two TDS. Then $\underline{a}_1$ and $\underline{a}_2$ are conjugate to each other. Furthermore the conjugation can be chosen so as to leave e fixed.

Proof. According to §2.3(b) we may find two S-triples each containing e as nil-positive element and which, respectively, are bases for $\underline{a}_1$ and $\underline{a}_2$. Corollary 3.6 then follows immediately from Theorem 3.6. Q.E.D.

3.7. Corollary 3.6 is actually just a special case of the following corollary. The set of all TDS in $\underline{g}$ breaks up into conjugate classes under the action of G. Concerning these classes we have

Corollary 3.7. The conjugate classes of TDS in $\underline{g}$ are in a natural one-one correspondence with the conjugate classes of non-zero nilpotent elements in $\underline{g}$. The correspondence is established by associating to the conjugate class of $\underline{a}$, $\underline{a}$ TDS in $\underline{g}$, the conjugate class of any non-zero nilpotent element in $\underline{a}$.

That is, two TDS $\underline{a}_1$ and $\underline{a}_2$ are conjugate if and only if e_1 and e_2 are conjugate where $e_1 \in \underline{a}_1$, $e_2 \in \underline{a}_2$ and e_1, e_2 are non-zero nilpotent elements.

Proof. Follows immediately from §2.3(b), Theorem 3.4 and Corollary 3.6. Q.E.D.

4. Semi-simple elements and TDS. 1. We now consider the semi-simple elements of a TDS and take up questions of conjugacy. Let $\{x, e_+, e_-\}$ be an S-triple with x and e_+, respectively, as the neutral and nil-positive elements. We wish first to determine all S-triples which contain x (necessarily as neutral element). Let $\underline{a}$ be the TDS spanned by x, e_+ and e_-. By considering the adjoint representation of $\underline{a}$ on $\underline{g}$ it follows from §2.5 that the eigenvalues of ad x on $\underline{g}$ are integers and half-integers. In fact, recalling §2.5(f) and §2.5(i) if $\underline{g}_{\frac{p}{2}}$ is the eigenspace of ad x for the eigenvalue $\frac{p}{2}$ and if k is the maximal value of ad x then

$$\dim \underline{g}_{\frac{p}{2}} = \dim \underline{g}_{-\frac{p}{2}}$$

and

$$\underline{g} = \sum_{p=-2k}^{2k} \underline{g}_{\frac{p}{2}}$$

In this case, however, we have the additional relation

$$\left[\underline{g}_{\frac{p}{2}}, \underline{g}_{\frac{p'}{2}}\right] \subseteq \underline{g}_{\frac{p+p'}{2}} \qquad (4.1.1)$$

Since we are concerned with S-triples containing x interest focuses on $\underline{g}_1$ since any nil-positive element in an S-triple

containing x obviously belongs to $\underline{g}_1$. In particular $e_+ \in \underline{g}_1$ The question arises which other elements of $\underline{g}_1$ belong to S-triples containing x and whether such S-triples are conjugate to $\{x, e_+, e_-\}$. To settle this question we first consider $\underline{g}_0 = \underline{g}^x$.

It follows immediately from (4.1.1) that $\underline{g}^x$ is a Lie subalgebra. Let G^x be the subgroup of the adjoint group corresponding to the subalgebra $\underline{g}^x$. It also follows from (4.1.1) that each of the subspaces $\underline{g}_p$ is stable under the adjoint representation of $\underline{g}^x$ on $\underline{g}$. And hence these spaces must be stable under G^x. We are particularly interested in the action of G^x on $\underline{g}_1$.

4.2. Now for each element $e \in \underline{g}_1$ it follows from that

$$\text{ad } e : \underline{g}_0 \longrightarrow \underline{g}_1 \quad .$$

Let T_e be the restriction of ad e to $\underline{g}_0$. Our interest now centers on what will be shown to be an important subset of $\underline{g}_1$. Define

$$\hat{\underline{g}}_1 = \{e \in \underline{g}_1 \mid T_e \text{ maps } \underline{g}_0 \text{ onto } \underline{g}_1\}.$$

That is, $e \in \hat{\underline{g}}_1$ if and only if the rank of T_e equals the dimension of $\underline{g}_1$.

We now observe

Lemma 4.2A. <u>A necessary condition that an element</u> $e \in \underline{g}_1$ <u>be the nil-positive element of an S-triple containing</u> x <u>is that</u> $e \in \hat{\underline{g}}_1$. <u>In particular</u> $e_+ \in \hat{\underline{g}}_1$.

<u>Proof</u>. Indeed assume e and x belong to an S-triple. Let

$\underline{a}'$ be the TDS which contains e and x. If we apply §2.5(i) to the adjoint representation of $\underline{a}$ on $\underline{g}$ it follows that $\underline{g}_1$ is the range of ad e. On the other hand it is clear from (4.1.1) that ad $e(\underline{g}) \cap \underline{g}_1 = \text{ad } e(\underline{g}_0) \cap \underline{g}_1$. Hence T_e must map $\underline{g}_0$ onto $\underline{g}_1$. Q.E.D.

The following topological properties of $\hat{\underline{g}}_1$ are needed for the proof of Theorem 4.2 (Here we were inspired by the use of regular elements in the usual proof of the conjugacy of any two Cartan subalgebras.).

Lemma 4.2B. The set $\hat{\underline{g}}_1$ is an open, dense and connected subset of $\underline{g}_1$.

Proof. It follows from Lemma 4.2A that $\hat{\underline{g}}_1$ is not empty ($e_+ \in \hat{\underline{g}}_1$). Choose a basis of $\underline{g}_0$ and a basis of $\underline{g}_1$. For any $e \in \underline{g}_1$ set T_e^o equal to the dim $\underline{g}_0$ x dim $\underline{g}_1$ matrix determined by T_e and the given pair of bases. It is clear then that $\hat{\underline{g}}_1$ is the set of all $e \in \underline{g}_1$ such that at least one dim $\underline{g}_1$ x dim $\underline{g}_1$ minor of T_e^o is not zero. But it is an easily verified general fact that if F_j, $j = 1,2,\ldots,m$ are m non-zero polynomials on a complex vector V then the complement $\hat{V}$ to the set of common zeros in V of all the F_j is open, dense and connected. Indeed if $u \in \hat{V}$ and $v \in V$ then there are only a finite number of complex scalars λ such that $F_j^o(\lambda) = F_j(\lambda u + (1-\lambda)v)$ vanishes for at least one j. Q.E.D.

Now observe that $\hat{\underline{g}}_1$ is invariant under G^x. The major point in the proof of Theorem 3.4 is the observation contained in the following lemma.

Lemma 4.2C. *Let* $e \in \hat{\mathfrak{g}}_1$ *then the orbit* $G^x e$ *of* e *under the action of* G^x *is an open subset of* $\mathfrak{g}_1$ *(and hence of* $\hat{\mathfrak{g}}_1$*).*

Proof. The mapping $A \longrightarrow Ae$ of G^x into $\mathfrak{g}_1$ is analytic. Thus it suffices to show that the differential of this mapping carries the tangent space to G^x at 1 *onto* the tangent space of $\mathfrak{g}_1$ at e. But the image of the former, under the differential, when translated to the origin of $\mathfrak{g}_1$ is just the subspace $\operatorname{ad} \mathfrak{g}^x(e)$ of $\mathfrak{g}_1$. But $\mathfrak{g}^x = \mathfrak{g}_0$ and since $e \in \hat{\mathfrak{g}}_1$ this space coincides with $\mathfrak{g}_1$, by definition of $\hat{\mathfrak{g}}_1$. Thus $G^x e$ is open in $\mathfrak{g}_1$. Q.E.D.

We have shown $G^x e$ is open in $\hat{\mathfrak{g}}_1$ for any $e \in \hat{\mathfrak{g}}_1$. But for $e_1, e_2 \in \mathfrak{g}_1$, $G^x e_1$ and $G^x e_2$ are the same sets or else they are disjoint. But this fact taken together with Lemma 4.2B (the latter asserting the connectivity of $\hat{\mathfrak{g}}_1$) implies that there can be at most one orbit. That is, $\hat{\mathfrak{g}}_1$ is itself a single orbit of G^x.

But then recalling Lemma 4.2A and observing that x is fixed under the action of G^x we see that the following theorem has been proved.

Theorem 4.2. *Let* $x \in \mathfrak{g}$ *be the neutral element of an S-triple* $\{x, e_+, e_-\}$. *(See* §3.5.*) As in* §2.1 *let* $\mathfrak{g}^x$ *be the centralizer of* x *in* $\mathfrak{g}$ *and let* G^x *be the subgroup of* G *corresponding to* $\mathfrak{g}^x$.

Define

$$\mathfrak{g}_1 = \{e \in \mathfrak{g} \mid [x,e] = e\}.$$

Then

$$\operatorname{ad} e : \mathfrak{g}^x \longrightarrow \mathfrak{g}_1 \qquad (4.2.1)$$

for any $e \in \mathfrak{g}_1$.

Let $e \in \underline{g}$. Then e and x are, respectively, the nil-positive and neutral elements of an S-triple if and only if (1) $e \in \underline{g}_1$ and (2) the map (4.2.1) is onto. Moreover any two S-triples which contain x are conjugate to each other and the conjugation can be performed by an element in G^x.

In other words if

$$\hat{\underline{g}}_1 = \{e \in \underline{g}_1 \mid (4.2.1) \text{ is an onto map}\}$$

then $\hat{\underline{g}}_1$ is the conjugate class of e_+ under the action of G^x. That is

$$\hat{\underline{g}}_1 = G^x e_+ \quad .$$

Furthermore $\hat{\underline{g}}_1$ coincides with the set of all nil-positive elements taken from all S-triples having x as neutral element.

An immediate corollary to Theorem 4.2 is a result of Malcev given here as Corollary 4.2 below (see[10]). The result of Malcev provides the basis by which all the TDS in any complex simple Lie algebra have been classified (see [7], §8). This in turn has led to a classification of more general Lie subalgebras of a complex simple Lie algebra (see [7], Chap. IV).

We wish to remark that Theorem 4.2 is stronger than the theorem of Malcev in that it gives a complete and simple description of the set of all nil-positive elements which "go" with a given neutral element. It shows this set to be the complement of an algebraic variety in $\underline{g}_1$ and therefore the set is an open, dense and connected subset of $\underline{g}_1$. The proof of Theorem 4.2 is entirely different from the proof given by Malcev of Corollary 4.2.

The finding of the former was motivated partly because the latter is not clear to us.

Corollary 4.2 (Malcev). Two TDS in $\underline{g}$ are conjugate if and only if any mono-semisimple element of one is conjugate to any mono-semisimple element of the other (see [10]).

4.3. Now, continuing with the notation of §4.1 and §4.2, the subset $\hat{\underline{g}}_1$ in $\underline{g}_1$ is of course only a part of the conjugate class of e_+ in $\underline{g}$. It does not seem likely that one can give a simple description of the entire class. Nevertheless, it is easy to describe a part of this class which is yet larger than $\hat{\underline{g}}_1$ (see Theorem 4.3, Theorem 4.3 is required for the proof of Theorem 5.3) and we shall do this now.

For any $t = 0,1,\ldots,2k$ let

$$\underline{n}_{\frac{t}{2}} = \sum_{t=p}^{2k} \underline{g}_{\frac{p}{2}} .$$

It is clear from (4.1.1) that $\underline{n}_{\frac{t}{2}}$ is a Lie subalgebra of $\underline{g}$. Let $N_{\frac{t}{2}}$ be the subgroup of G corresponding to $\underline{n}_{\frac{t}{2}}$. We how have, in terms if the preceding notation,

Theorem 4.3. The orbits N_0x and N_0e_+ of e_+ and x under the action of the group N_0 are as follows:

$$N_0x = x + \underline{n}_{\frac{1}{2}}$$

and

$$N_0e_+ = \hat{\underline{g}}_1 + \underline{n}_{\frac{3}{2}} .$$

In fact G^x *is a subgroup of* N_0 *and* $N_{\frac{1}{2}}$ *is a normal subgroup of* N_0. *Moreover, the elements of* $N_{\frac{1}{2}}$ *are unipotent linear transformations of* g. *Furthermore* N_0 *can be written as a semi-direct product*

$$N_0 = N_{\frac{1}{2}} G^x$$

(*every element* $C \in N_0$ *is uniquely written* $C = AA'$ *where* $A \in N_{\frac{1}{2}}$ $A' \in G^x$) *and the correspondence*

$$A \longrightarrow Ax$$

sets up a one- one mapping of $N_{\frac{1}{2}}$ *onto* $x + \underline{n}_{\frac{1}{2}}$ *while the correspondence*

$$A \longrightarrow Ae_+$$

defines a mapping of N_0 *onto* $e_+ + \underline{n}_{\frac{3}{2}}$.

Proof. Since $\underline{n}_0 = \underline{g}^x + \underline{n}_{\frac{1}{2}}$ is a semi-direct sum (that is, $[\underline{g}^x, \underline{n}_{\frac{1}{2}}] \subseteq \underline{n}_{\frac{1}{2}}$) it follows that every element $C \in N_0$ can be written, $C = AA'$, where $A \in N_{\frac{1}{2}}$ and $A' \in G^x$. On the other hand if $A \in N_{\frac{1}{2}}$ and $y \in \underline{g}_{\frac{t}{2}}$ then $Ay = y+w$ where $\sum_{p=t+1}^{2k} \underline{g}_{\frac{p}{2}}$. This implies A is unipotent. It also implies that $N_{\frac{1}{2}} \cap G^x = (1)$ and hence N_0 is a semi-direct product of $N_{\frac{1}{2}}$ and G^x. Now observe that $\underline{n}_{\frac{1}{2}}$ is stable under ad x and ad x is non-singular on $\underline{n}_{\frac{1}{2}}$. The proof that the correspondence $A \longrightarrow Ax$ sets up a one-one mapping of $N_{\frac{1}{2}}$ onto $x + \underline{n}_{\frac{1}{2}}$ then proceeds in essentially the same way as the proof of Theorem 3.6 (Recall

that $\left[\underline{n}_{\frac{t}{2}}, \underline{n}_{\frac{t'}{2}}\right] \subseteq \underline{n}_{\frac{t+t'}{2}}$ and see the note following Theorem 3.6) and we shall not repeat it. On the other hand one cannot claim that the mapping of $N_{\frac{1}{2}}$ into $e_+ + \underline{n}_{\frac{3}{2}}$ defined by the correspondence $A \longrightarrow Ae$ is one-one. This is because ad e_+ annihilates non-zero elements in $\underline{n}_{\frac{1}{2}}$. But ad e_+ maps $\underline{g}_{\frac{p}{2}}$ into $\underline{g}_{\frac{p}{2}+1}$ and applying 2.5(i) to the adjoint representation of $\underline{a}$ on $\underline{g}$ it follows that

$$\text{ad } e_+ : \underline{n}_{\frac{1}{2}} \longrightarrow \underline{n}_{\frac{3}{2}}$$

is an onto mapping. Proceeding then in a manner similar to the proof of Theorem 3.6 it follows easily that $N_{\frac{1}{2}}$ is mapped onto $e_+ + \underline{n}_{\frac{3}{2}}$ by the correspondence $A \longrightarrow Ae_+$. Q.E.D.

As we have remarked, Corollary 4.2 has provided the basis by which the conjugate classes of TDS have been classified (See [7], § 8). That is if x is a mono-semisimple element of a TDS we know that x is a semi-simple element of $\underline{g}$ and that the eigenvalues of ad x are real (in fact are integers and half-integers). To classify the conjugate classes of TDS it suffices then by Corollary 4.2 to find a fundamental domain for the action of G on the set of semi-simple x in $\underline{g}$ such that ad x has real eigenvalues and determine which in the domain are mono-semisimple elements of TDS. The Weyl chamber is such a domain and we shall presently consider it.

(In view of Corollary 3.7 for the purposes of classifying conjugate classes of TDS it might be well to look for natural representative elements for the conjugate classes of nilpotent elements. To my knowledge no such investigation has been undertaken).

5. The principle TDS. 1. Let $\underline{h} \subsetneq \underline{g}$ be a Cartan subalgebra which we shall assume is fixed once and for all. Let Δ be the set of roots with respect to $\underline{h}$ and let e_φ, $\varphi \in \Delta$ be representative root vectors so that we have usual direct sum decomposition

$$\underline{g} = \underline{h} + \sum_{\varphi \in \Delta} (e_\varphi) .$$

Since B is non-singular on $\underline{h}$ we may assume Δ is embedded in $\underline{h}$. That is, if $\varphi \in \Delta$ then φ may be identified with an element in $\underline{h}$ by the relation

$$[x, e_\varphi] = (x, \varphi) e_\varphi \quad .$$

for all $x \in \underline{h}$. One knows then that if e_φ and $e_{-\varphi}$ are normalized only in so far as

$$(e_\varphi, e_{-\varphi}) = 1, \qquad (5.1.0)$$

as we shall assume, then

$$[e_\varphi, e_{-\varphi}] = \varphi \qquad (5.1.1)$$

Now let $\underline{h}^*$ be the real linear space in $\underline{h}$ spanned by the roots. One knows that

$$\underline{h} = \underline{h}^* + i\underline{h}^*$$

is a real direct sum. We recall that B is positive definite on $\underline{h}^*$ (see e.g. [12], p. 10-04). In particular then, (x, φ) is real for all $x \in \underline{h}^*$ at all $\varphi \in \Delta$. This means $\underline{h}^*$ can be characterized as the set of all elements $x \in \underline{h}$ such that ad x has real eigenvalues.

Hereafter, if Y is any subset of $\underline{h}$ we will let $Y^* = Y \cap \underline{h}^*$.

Now for any $\varphi \in \Delta$ let

$$\underline{h}[\varphi] = \{x \in \underline{h} \mid (\varphi, x) = 0\} .$$

Then, clearly, if R is the set of all regular elements in $\underline{h}$

$$R^* = \underline{h} - \bigcup_{\varphi \in \Delta} \underline{h}[\varphi] .$$

The connected components of R^* are called open Weyl chambers. Now assume a lexicographical ordering is given in $\underline{h}^*$. Let Δ^+ (resp. Δ^-) be the set of positive roots (resp. negative roots). The ordering distinguishes a particular open Weyl chamber D° which can be defined by

$$D^\circ = \{x \in \underline{h}^* \mid (\varphi, x) > 0 \text{ for all } \varphi \in \Delta^+\} .$$

The closure D of D° is called a Weyl chamber. It can be defined in the same way as D° except that one replaces the strict inequality $>$ by the inequality $\geq$. The set D is a fundamental domain for the action of G on the set of all semi-simple elements y in $\underline{g}$ such that ad y has real eigenvalues. That is, if y is such an element there exists one and only one ([12], p. 16-08 and [7], Lemma 8.2) element x in D which is conjugate to y.

Now let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ be the set of simple positive roots. The simple positive roots form a basis of $\underline{h}$ and for every root φ, upon writing

$$\varphi = \sum_{i=1}^{\ell} n_i \alpha_i ,$$

one knows that the coefficients n_i are integers which are all non-negative or all non-positive according as $\varphi \in \Delta^+$ or $\varphi \in \Delta^-$. We define the order $o(\varphi)$ of φ by letting

$$o(\varphi) = \sum_{i=1}^{\ell} n_i \quad . \tag{5.1.2}$$

Obviously, if φ_1, φ_2 and $\varphi_1 + \varphi_2 \in \Delta$ then

$$o(\varphi_1 + \varphi_2) = o(\varphi_1) + o(\varphi_2) \tag{5.1.3}$$

Let ϵ_i, $i = 1,2,\ldots,\ell$ be the dual basis in $\underline{h}$ to the α_j. Since

$$(\epsilon_i, \alpha_j) = \delta_{ij} \tag{5.1.4}$$

it is clear that $\epsilon_i \in D$. More generally if

$$x = \sum_{i=1}^{\ell} a_i \epsilon_i \tag{5.1.5}$$

then $x \in D$ if and only if $a_i \geq 0$, $i = 1,2,\ldots,\ell$. Indeed

$$(x, \alpha_i) = a_i \tag{5.1.6}$$

Now assume $x \in D$ is the neutral element of an S-triple $\{x, e_+, e_-\}$. Since the eigenvalues of ad x are integral multiples of $\frac{1}{2}$ it follows from that in the expansion (5.1.5) $a_i = \frac{1}{2} m_i$ for some non-negative integer m_i. On the other hand it is not porrible for a_i to be greater than 1. Indeed since ad $x(e_-) = -e_-$

it is clear that

$$e_- = \sum_{\varphi \in \Delta^-} c_\varphi e_\varphi$$

where we emphasize the summation is over negative roots. Hence it follows from §2.5(d) (with signs reversed. Also see Lemma 3.4) that if $a_i > 1$, ad $e_-(e_{\alpha_i})$ is a non-zero eigenvector of ad x with the positive eigenvalue $a_i - 1$. But then since $x \in D$ this implies $[e_-, e_{\alpha_i}]$ is a sum of root vectors for positive roots. This, however, clearly contradicts the simplicity of α_i. We have proved then that $a_i = 0, \frac{1}{2}$, or 1 for $i = 1,2,\ldots,\ell$. More than this we have (see [7], Theorem 8.3).

Lemma 5.1. *Let the Weyl chamber D and the basis ϵ_1, $i = 1,2,\ldots,\ell$ of h be given as above. Assume that $x \in D$ is a mono-semisimple element of a TDS. Then*

$$x = \sum_{i=1}^{\ell} a_i \epsilon_i$$

where for each $i = 1,2,\ldots,\ell$, we have $a_i = 0, \frac{1}{2}$, or 1. Furthermore if $x_1, x_2, \ldots, x_b \in D$ is the set (ordered) of all elements in D which happen to be mono-semisimple elements in at least one TDS (so that $b < 3^\ell$) then there are exactly b conjugate classes of TDS in g. In fact if x_i is a mono-semisimple element of the TDS a_i for $i = 1,2,\ldots, b$ then the subalgebras a_i, $i = 1,2,\ldots,b$ are representatives of the conjugate classes.

For each simple Lie algebra g Dynkin lists [7] which among the $3^\ell - 1$ choices are indeed mono-semisimple elements of a TDS.

5.2. We retain the notation of Lemma 5.1. Among the elements x_j, $j = 1,2,\ldots,b$ there is a distinguished one. This is the case when $a_i = 1$, $i = 1,2,\ldots, \ell$. That is, for one of the x_j all the a_i (in (5.15)) take the maximal possible value.

Lemma 5.2. *Let* D *and* ϵ_i, $i = 1,2,\ldots,\ell$ *be as defined in* 5.1. *Let* $x_o \in D$ *be given by*

$$x_o = \sum_{i=1}^{\ell} \epsilon_i \qquad (5.2.1)$$

Then x_o *is a mono-semisimple element of a* TDS.

Proof. Since the elements of α_i form a basis of h^*, x_o may be uniquely written

$$x_o = \sum_{i=1}^{\ell} r_i \alpha_i \ .$$

Now let c_i, $i = 1,2,\ldots,\ell$ be any ℓ non-zero complex numbers.

Define

$$e_o = \sum_{i=1}^{\ell} c_i e_{\alpha_i} \qquad (5.2.2)$$

and

$$f_o = \sum_{i=1}^{\ell} r_i / c_i \, e_{-\alpha_i} \ . \qquad (5.2.3)$$

Clearly $(\alpha_i, x_o) = 1$ for all i. Thus

$$[x_o, e_o] = e_o$$

and

$$[x_o, f_o] = -f_o \ .$$

On the other hand since $\alpha_i - \alpha_j$ is never a root it follows from (5.1.1) that

$$[e_o, f_o] = x_o$$

so that $\{x_o, e_o, f_o\}$ form a S-triple. Q.E.D.

Let x_o, e_o and f_o be as defined in the proof of Lemma 5.2. Let $\underline{a}_o$ be the TDS spanned by these vectors. The TDS $\underline{a}_o$ or any conjugate TDS is called (originally in [6]) a principle TDS of $\underline{g}$. It and some of its properties were discovered by Dynkin and de Siebenthal (See [6] and [13]). We shall call the S-triple (or any of its conjugates) $\{x_o, e_o, f_o\}$ a principle S-triple. One of the first properties we observe about the neutral element of a principle S-triple is that it is regular. In fact $x_o \in \underline{h}^*$ and for any $\varphi \in \Delta$ clearly

$$(x_o, \varphi) = o(\varphi) \neq 0 \tag{5.2.4}$$

(see (5.1.2)). We shall call x_o or any element of $\underline{g}$ conjugate to x_o a principle regular element of $\underline{g}$. Any element of the conjugate class of e_o will be called a principle nilpotent element of $\underline{g}$. We reserve the subscript $_o$ for quantities designated as principle.

One of the most significant ways in which a principle TDS is distinguished among all TDS is in regard to its adjoint representation on $\underline{g}$.

For any TDS $\underline{a}$ of $\underline{g}$ let $n(\underline{a})$, $n^E(\underline{a})$ and $n^O(\underline{a})$ designate, respectively, the number of irreducible components occurring in the complete reduction of the adjoint representation of $\underline{a}$ on $\underline{g}$, the number having even dimension and the number having odd dimension.

Theorem 5.2. *Let* $\underline{a}$ *be any* TDS *of* $\underline{g}$. *Then* $n(\underline{a}) \geq \ell$. *Furthermore* $\underline{a}$ *is principle if and only if* $n(\underline{a}) = \ell$.

Proof. Let $\{x,e,f\}$ be an S-triple whose linear span is $\underline{a}$. In fact employing the notation of § 5.1 we may assume $x \in D \subseteq \underline{h}$. Since $\underline{h}$ is contained in $\underline{g}^x$ (see § 2.1) obviously $\dim \underline{g}^x \geq \ell$. But now according to 2.5(g) we have $n^0(\underline{a}) = \dim \underline{g}^x$. Thus

$$n(\underline{a}) \geq n^0(\underline{a}) = \dim \underline{g}^x \geq \ell \qquad (5.2.5)$$

which proves the first part of Theorem 5.2. Now if $\underline{a}$ is principle x is regular and hence $\dim \underline{g}^x = \ell$. On the other hand in this case all the eigenvalues of ad x are integers (see (5.1.2)) so that $n^E(\underline{a}) = 0$ (see §2.5(e)). Thus if $\underline{a}$ is principle $n(\underline{a}) = n^0(\underline{a}) = \ell$. Now, conversely, assume $n(\underline{a}) = \ell$. But then by (5.2.5) $\dim \underline{g}^x = \ell$ which implies x is regular. Writing $x = \sum_{i=1}^{\ell} a_i \epsilon_i$ as in § 5.1, it follows then that $a_i = \frac{1}{2}$ or 1 for all i. But since $n^0(\underline{a}) = n(\underline{a})$, that is, $n^E(\underline{a}) = 0$, this means (see § 2.5(e)) the eigenvalues of ad x are all integral. Thus $a_i \neq \frac{1}{2}$ by (5.1.6) and hence $x = \sum_{i=1}^{\ell} \epsilon_i$ so that $\underline{a}$ is principle by Corollary 4.2.

Q.E.D.

As a corollary of the proof we have

Corollary 5.2. *Let* $\underline{a}$ *be any* TDS *of* $\underline{g}$ *then* $n^0(\underline{a}) \geq \ell$. *Furthermore* $n^0(\underline{a}) = \ell$ *if and only if the mono-semisimple elements of* $\underline{a}$ *are regular in* $\underline{g}$.

In case $\underline{a}$ *is principle* $n^E(\underline{a}) = 0$.

5.3. The following corollary of Theorem 5.2 distinguishes the principle nilpotent elements in the set of all nilpotent

elements in $\underline{g}$. The proof follows immediately from §2.5(d) and Theorem 5.2.

Corollary 5.3. *Let e be a nilpotent element in $\underline{g}$ then* $\dim \underline{g}^e \geq \ell$ *and* $\dim \underline{g}^e = \ell$ *if and only if e is principle nilpotent.*

Now let $\{x_o, e_o, f_o\}$ be the principle S-triple defined as in §5.2. We will now apply the theory of §4.1 to this S-triple.

First of all since the eigenvalues of ad x_o are integral we observe that $\underline{g}_{\frac{p}{2}} = 0$. whenever $\frac{p}{2}$ is not an integer and for j a non-zero integer

$$\underline{g}_j = \sum_{o(\varphi)=j} (e_\varphi) \qquad (5.3.1)$$

and

$$\underline{g}_0 = \underline{h} \quad .$$

We will write $\underline{n}$ for $\underline{n}_{\frac{1}{2}} = \underline{n}_1$ and $\underline{s}$ for $\underline{n}_0$. Then

$$\underline{n} = \sum_{\varphi \in \Delta^+} (e_\varphi)$$

(as is well known) is a maximal Lie subalgebra of nilpotent elements and $\underline{s} = \underline{h} + \underline{n}$ is a maximal solvable Lie subalgebra of $\underline{g}$. Let H, N and S be the subgroups of G corresponding to $\underline{h}$, $\underline{n}$ and $\underline{s}$.

Recalling that

$$e_o = \sum_{i=1}^{\ell} c_i e_{\alpha_i}$$

where $c_i \neq 0$, $i = 1,2,\ldots,\ell$ it follows from Theorems 4.2 and 4.3 that in this case

$$He_o = \hat{g}_1 = \{e \in g_1 \mid e = \sum_{i=1}^{\ell} a_{\alpha_i} e_{\alpha_i} \text{ where } a_{\alpha_i} \neq 0,\ i=1,2,\ldots,\ell\}$$

and (since $n_{\frac{3}{2}} = n_2$)

$$Se_o = \hat{g}_1 + n_2 = \{e \in n \mid e = \sum_{\varphi \in \Delta^+} a_\varphi e_\varphi \text{ where } a_\varphi \neq 0 \text{ when } \varphi = \alpha_i,\ i = 1,2,\ldots,\ell\}. \tag{5.3.2}$$

We emphasize that (5.3.2) above implies that "almost all" the elements in the maximal subalgebra of nilpotent elements n are principle nilpotent (and hence lie in a single conjugate class).

We wish to prove now that $S\ e_o$ contains every principle nilpotent in n, that is, we have the following simple characterization of those elements in n which are principle nilpotent.

Theorem 5.3. *Let $n \subseteq g$ be the maximal Lie subalgebra of nilpotent elements given by*

$$n = \sum_{\varphi \in \Delta^+} (e_\varphi)$$

Let $e \in n$,

$$e = \sum_{\varphi \in \Delta^+} a_\varphi e_\varphi$$

be arbitrary. Then e is principle nilpotent if and only if $a_\varphi \neq 0$ for $\varphi = \alpha_i$, $i = 1,2,\ldots,\ell$.

Proof. By (5.3.2) if e satisfies the condition states in Theorem 5.3 then e is principle nilpotent. Conversely, let $e \in n$ be principle nilpotent and assume $a_{\alpha_j} = 0$, for some j.

Assume first that g is simple. Let $\psi \in \Delta$ by the highest root. We recall two basic facts about the highest root. One,

upon writing $\psi = \sum_{i=1}^{\ell} q_i \alpha_i$ the integers q_i satisfy

$$q_i \geq 1 \tag{5.3.3}$$

and

$$o(\psi) > o(\varphi) \tag{5.3.4}$$

for any $\varphi \in \Delta$, $\varphi \neq \psi$. Indeed both of these facts are immediate consequences of the following single fact: If $\varphi = \sum_{i=1}^{\ell} t_i \alpha_i$ is any root then

$$t_i \leq q_i \tag{5.3.5}$$

for $i = 1,2,\ldots,\ell$. Finally (5.3.5) is a consequence of well known facts in representatiin theory; in fact, the adjoint representation of $\underline{g}$ is irreducible and by definition of ψ, e_ψ is the weight vector belonging to the highest weight, ψ. One knows then that e_φ, for any $\varphi \in \Delta$, is obtained by applying polynomials (non-commutative) in the operators ad $e_{-\alpha_i}$, $i = 1,2,\ldots,\ell$ to e_ψ. This proves (5.3.5). Let

$$q = \sum_{i=1}^{\ell} q_i = o(\psi).$$

Now let $\underline{a}_o$ be as in §5.2 and consider the adjoint representation of $\underline{a}_o$ on $\underline{g}$. It follows immediately from §2.5(d) and (5.3.4) that (1) e_ψ lies in an irreducible component $\underline{b} \subseteq \underline{g}$, (2) dim $\underline{b} = 2q + 1$, (3) $e_{-\psi} \in \underline{b}$ and (4) any other irreducible has dimension less than dim $\underline{b}$. A direct consequence of these facts is that

$$(\text{ad } e_o)^{2q+1} = 0$$

and

$$(\text{ad } e_o)^{2q}(e_{-\psi}) = a\, e_{\psi}$$

where $a \neq 0$.

Now since e is conjugate to e_o it follows that $(\text{ad } e)^{2q} \neq 0$. On the other hand since $e \in \underline{n}$ it follows that for any $\varphi \in \Delta$ we can write

$$(\text{ad } e)^{2q}(e_{\varphi}) = \sum_{\xi \in \Delta} b_{\xi}\, e_{\xi} \qquad (5.3.6)$$

and, moreover, $b_{\xi} \neq 0$ implies

$$o(\xi) \geq o(\varphi) + 2q \quad .$$

But this together with (5.3.4) implies that (5.3.6) vanishes for all $\varphi \neq -\psi$. Thus $(\text{ad } e)^{2q}(e_{-\psi}) \neq 0$ and in fact using (5.3.4) once more

$$(\text{ad } e)^{2q} e_{-\psi} = a' e_{\psi}$$

for some non-zero scalar a'. Now write $e = e_1 + e_2$ where $e_1 \in \underline{g}_1$ and $e_2 \in \underline{n}_2$.

Expanding $(\text{ad } e)^{2q} = (\text{ad } e_1 + \text{ad } e_2)^{2q}$ it is clear again from (5.3.4) that

$$(\text{ad } e)^{2q}(e_{-\psi}) = (\text{ad } e_1)^{2q}(e_{-\psi}) = a' e_{\psi} \quad .$$

But for any i writing

$$(\text{ad } e_1)^{i}(e_{-\psi}) = \sum_{\xi \in \Delta} b'_{\xi}\, e_{\xi}$$

it follows that since $a_{\alpha_j} = 0$ one can have $b'_{\xi} \neq 0$ only when upon writing

$$\xi = \sum_{i=1}^{\ell} t_i \alpha_i$$

we have $t_j = -q_j$. In particular setting $i = 2q$ this means

$q_j = -q_j$ or $q_j = 0$. This contradicts (5.3.3) and hence Theorem 5.3 is proved when $\underline{g}$ is simple. The general case follows immediately upon writing $\underline{g}$ as a direct sum of its simple ideals. The component of e in each ideal is necessarily principle nilpotent in that ideal by Corollary 5.3. Q.E.D.

5.4. A theorem proved in [1] (See [1], Remark p. 66.) asserts that every solvable subalgebra $\underline{s}_1$ of $\underline{g}$ is conjugate to a subalgebra $\underline{s}_1'$ of $\underline{s}$. If in addition it is assumed that the elements of $\underline{s}_1$ are all nilpotent (hence $\underline{s}_1$ is a nilpotent Lie algebra by Engel's theorem) it is clear then that $\underline{s}_1' \subseteq \underline{n}$. In particular it follows then that every nilpotent element $e \in \underline{g}$ is conjugate to some element $e' \in \underline{n}$. Actually we don't need the result of [1] referred to above to prove this. For completeness, we observe that this follows from Lemma 5.1.

Lemma 5.4. *Any nilpotent element* $e \in \underline{g}$ *is conjugate to an element* $e' \in \underline{n}$.

Proof. Applying Theorem 3.4 it suffices only to show that the nil-positive element of any S-triple containing x_j (using the notation of Lemma 5.1) as neutral element lies in $\underline{n}$. But this is clear since $x_j \in D$. That is, for $\varphi \in \Delta, (x_j, \varphi) = 1$ implies $\varphi \in \Delta^+$. Q.E.D.

As a corollary to Theorem 5.3, its proof, and Lemma 5.4 we obtain another characterization of principle nilpotent elements in case $\underline{g}$ is simple.

Corollary 5.4. *Assume* $\underline{g}$ *is simple. Let* ψ *be the highest*

root and let $q = o(\psi)$. Let $e \in g$ be any nilpotent element. Then e is principle nilpotent if and only if $(\text{ad } e)^{2q} \neq 0$. However, if e is principle nilpotent $(\text{ad } e)^{2q+1} = 0$.

5.5. Corollary 3.7 sets up a natural one-one relation between the conjugate classes of nilpotent elements and the conjugate classes of TDS. It is clear that in this correspondence the class of principle nilpotent elements corresponds to the class of principle TDS. Regarding the latter as distinctive among all conjugate classes of TDS will be given further justification then when it is shown that the former is distinctive among conjugate classes of nilpotent elements. The following corollary shows this very clearly.

Corollary 5.5. The set of principle nilpotent elements (a conjugate class of the adjoint group G) in g forms an open, dense and connected subset of the set of all nilpotent elements in g.

Proof. Openness follows easily from Corollary 5.3 by choosing a basis of g and considering the $(n-\ell)\times(n-\ell)$ minors of the matrix defined by ad e, e nilpotent, with respect to the basis. Denseness follows from Theorem 5.3 and Lemma 5.4. Connectivity is immediate also since the set of principle nilpotent elements is an orbit of the group G.

5.6. Principle nilpotent elements behave like the "regular" elements in the set of all nilpotent elements. That is, one can make a strong case for the following analogy: The set of principle nilpotent elements is to the set of all nilpotent elements as

the set of all regular elements is to the set of all semi-simple elements. We cite for example Corollary 5.3 and Corollary 5.5. In this analogy, between the semi-simple elements and the nilpotent elements, the Cartan subalgebra clearly corresponds to the maximum Lie subalgebra of nilpotent elements (see Lemma 5.4 §2.1. Also all maximum Lie subalgebras of nilpotent elements are conjugate - see §5.4). One knows that a regular element can be characterized by the property that it lies in one and only one Cartan subalgebra. Corollary 5.6 asserts that also in this regard the analogy still holds.

Corollary 5.6. Let $e \in g$ be nilpotent. (One knows that e can be embedded in at least one maximal Lie subalgebra of nilpotent elements.) Then e is principle nilpotent if and only if e lies in one and only one maximal Lie subalgebra of nilpotent elements of g.

Proof. By Lemma 5.4 we may assume $e \in \underline{n}$ (using the notation §5.4). Assume that e is not principle. We will prove e is contained in a second (different) maximal Lie subalgebra of nilpotent elements $\underline{n}'$. Now we may write

$$e = \sum_{\varphi \in \Delta^+} a_\varphi e_\varphi \quad .$$

By Theorem 5.3 since e is not principle $a_{\alpha_i} = 0$ for some value of i. But now it is known (and easy to verify) that

$$\underline{n}' = \sum_{\substack{\varphi \in \Delta^+ \\ \varphi \neq \alpha_i}} (e_\varphi) + (e_{-\alpha_i})$$

is a maximal Lie subalgebra of nilpotent elements of g. (In fact $\underline{n}$ is carried

onto $\underline{n}'$ by any element of G which (1) leaves $\underline{h}$ invariant and (2) whose restriction to $\underline{h}$ is the reflection R_{α_i} - see §7.1). Obviously $e \in \underline{n}'$.

Now assume e is principle. In fact we may take $e = e_o$ where we use the notation of §5.2. Assume $e_o \in \underline{n}'$ where $\underline{n}' = A\underline{n}$, $A \in G$. We shall use the prime(') on previous notation to indicate the effect of conjugation by A. Applying Theorem 5.3 to $\underline{n}'$ it follows that

$$e_o = \sum_{\varphi \in \Delta} + a_\varphi e'_\varphi$$

where $a_\varphi \neq 0$ for $\varphi = \alpha_i$, $i = 1,2,\ldots,\ell$. Now by Theorem 4.3 there exists $A_1 \in N'$ such that

$$A_1 e_o = \sum_{i=1}^{\ell} a_{\alpha_i} e'_{\alpha_i} \quad .$$

But then since x'_o $(=Ax_o)$ and $A_1 e_o$ are clearly the neutral and nil-positive elements of an S-triple (see proof of Lemma 5.2) it follows from Theorem 3.6 that

$$A_1 x_o \in x'_o + \underline{g}^{A_1 e_o} \quad .$$

However, since x'_o is regular upon applying §2.5(d) and (5.1.2) in the case of this S-triple it follows that $\underline{g}^{A_1 e_o} \subseteq \underline{n}'$. But then by Theorem 4.3 there exists $A_2 \in N'$ such that $A_2 A_1 x_o = x'_o$. But now since x_o is regular we must have $A_2 A_1 \underline{h} = \underline{h}'$. But this of course implies $\underline{h} \subseteq \underline{s}'$ because $A_1^{-1} A_2^{-1} \in S'$. But this means $\underline{s}'$ and hence $\underline{n}' = [\underline{s}', \underline{s}']$ are stable under ad $\underline{h}$. But then $\underline{n}'$

must contain and in fact must be spanned by root vectors associated with $\underline{h}$ (since obviously $\underline{n}' \cap \underline{h} = 0$). But $e_o = \sum_{i=1}^{\ell} c_i e_{\alpha_i} \in \underline{n}'$ and $c_i \neq 0$. Thus $e_{\alpha_i} \in \underline{n}'$, $i = 1,2,\ldots,\ell$. Hence since the e_{α_i} generate $\underline{n}$ it follows that $\underline{n} \subseteq \underline{n}'$. But then of course $\underline{n} = \underline{n}'$. Q.E.D.

5.7. We continue with previous notation.

Now consider $\underline{g}^{e_o}$, the kernel of ad e_o. By Corollary 5.3

$$\dim \underline{g}^{e_o} = \ell .$$

In the special case when $\underline{g}$ is the set of all $(\ell+1)\times(\ell+1)$ complex matrices of trace zero one sees easily that $\underline{g}^{e_o}$ is a commutative Lie algebra (of nilpotent matrices). This and other evidence suggested to us that perhaps $\underline{g}^{e_o}$ is commutative in the general case. However, since among other things we were unable to construct a "useable" basis of $\underline{g}^{e_o}$ we could not settle the question using purely algebraic methods. Nevertheless it is true that $\underline{g}^{e_o}$ is commutative (Corollary 5.8) is the general case. The proof which we have found is very simple but uses limit arguments. It was Corollary 5.8 which originally suggested the validity of Theorem 6.7.

Theorem 5.7 or Corollary 5.7 may be regarded as a generalization of the fact that a Cartan subalgebra of a semi-simple Lie algebra is commutative.

Theorem 5.7. *Let $\underline{g}$ be a complex semi-simple Lie algebra of rank ℓ. Let $y \in \underline{g}$ be arbitrary. Then $\underline{g}^y$ contains an ℓ dimensional commutative Lie subalgebra.*

Proof. It is well known that the set of regular elements in $\underline{g}$ is dense in $\underline{g}$. Thus we may find a sequence y_n', $n = 1,2,\ldots$ of regular elements converging to y.

Now consider the Grassman manifold of all ℓ planes in $\underline{g}$. This, of course, is compact and hence we may find subsequence y_n of the sequence y_n' with the property $y_n \longrightarrow y$ and the Cartan subalgebras $\underline{g}^{y_n}$ converge to an ℓ-plane $\underline{u}$ in the Grassman manifold. Now if w_i, $i = 1,2,\ldots,\ell$ is any basis of $\underline{u}$ we may find elements $w_i^n \in \underline{g}$, $n = 1,2,\ldots$, $i = 1,2,\ldots,\ell$ such that $w_i^n \in \underline{g}^{y_n}$ and $w_i^n \longrightarrow w_i$ as $n \longrightarrow \infty$ for $i = 1,2,\ldots,\ell$. Since $[y_n, w_i^n] = 0$ it follows immediately by taking the limit that $\underline{u} \subseteq \underline{g}^y$. But $[w_i^n, w_j^n] = 0$. Again taking the limit this obviously implies $\underline{u}$ is commutative. Q.E.D.

Corollary 5.7. Let $y \in \underline{g}$. Assume $\dim \underline{g}^y = \ell$. Then $\underline{g}^y$ is commutative.

5.8. Corollary 5.7 and Corollary 5.3 imply

Corollary 5.8. Let e be a principle nilpotent element in $\underline{g}$. Then $\underline{g}^e$ is commutative.

6. The principle element of G and the duality theorem. 1. We shall assume from now on that $\underline{g}$ is simple. The theorems to be proved are either true in the general semi-simple case or can be obviously modified to be true in that case. The extension from the simple case to the semi-simple case in any event is immediate. We consider only the former mainly for notational simplicity.

Recall what is meant by a compact form $\underline{k}$ of $\underline{g}$. This may be defined as a real Lie subalgebra of $\underline{g}$ with the property

(1) $\underline{g} = \underline{k} + i\underline{k}$

is a real direct sum. (That is, $\underline{k}$ is just a "real form" of $\underline{g}$ and

(2) The restriction of B to $\underline{k}$ is negative definite.

One immediate consequence of (2) is

(a) every element of $\underline{k}$ is semi-simple. Next we recall some facts in the Cartan subalgebra theory of $\underline{k}$ (which is somewhat different from that of $\underline{g}$).

(b) If $\underline{t}$ is a Cartan subalgebra of $\underline{k}$ then $\underline{h}_1 = \underline{t} + i\underline{t}$ is a Cartan subalgebra of $\underline{g}$ and $\underline{h}_1^* = i\underline{t}$.

(c) A Lie subalgebra $\underline{t}$ of $\underline{k}$ is a Cartan subalgebra if and only if it is maximal commutative.

An immediate consequence of (c) which we shall soon use is

(d) Any commutative Lie subalgebra $\underline{u}$ of $\underline{k}$ can be embedded in a Cartan subalgebra of $\underline{k}$.

Corresponding to the adjoint operation (conjugate transpose) in the space of matrices we introduce a *-operation in $\underline{g}$ by defining for any $z \in \underline{g}$

$$z^* = x - iy$$

where z is written

$$z = x + iy$$

for $x, y \in i\underline{k}$.

Generalizing the notion of a normal matrix or normal operator we will say $z \in \underline{g}$ is normal with respect to $\underline{k}$ whenever

$$[z,z^*] = 0$$

Writing $z = x + iy$, $x,y \in i\underline{k}$ it is clear that z is normal if and only if $[x,y] = 0$.

It is well known of course that a normal matrix is diagonizable (semi-simple). The following generalization of this fact (and also of (a)) is a useful criterion for semi-simplicity in $\underline{g}$.

Lemma 6.1. *Let z, an element of $\underline{g}$, be normal with respect to a compact form $\underline{k}$ of $\underline{g}$. Then z is a semi-simple element of $\underline{g}$.*

Proof. We may write $z = x + iy$ where $ix, iy \in \underline{k}$. But now since z is normal $[ix,iy] = 0$. Thus by (d) there exists a Cartan subalgebra $\underline{t}$ of $\underline{k}$ which contains ix and iy. But by (b) $\underline{h}_1 = \underline{t} + i\underline{t}$ is a Cartan subalgebra of $\underline{g}$. But then $z \in \underline{h}_1$ and hence z is semi-simple. Q.E.D.

When the root vectors, relative to a Cartan subalgebra, are suitably normalized (Weyl's normal form) Weyl has given a basis of a compact form of $\underline{g}$ in terms of these root vectors and the Cartan subalgebra. In previous sections we required no other normalization of the e_φ, $\varphi \in \Delta$, other than (5.1.0). We will assume from here on, unless specified otherwise (in Theorem 8.4), that the root vectors are normalized into Weyl's normal form, with the exception that $(e_\varphi, e_{-\varphi}) = 1$ is retained in preference to $(e_\varphi, e_{-\varphi}) = -1$. (In such a normal form one may choose the root vectors e_{α_i}, $i = 1,2,\ldots,\ell$, arbitrarily

and hence we need not regard e_0 as having been altered). Then one knows that the linear span, with real coefficients, of $i\underline{h}^*$ and the vectors $e_\varphi - e_{-\varphi}$, $ie_\varphi + ie_{-\varphi}$ for all $\varphi \in \Delta^+$ is a compact form $\underline{k}$ of $\underline{g}$. (See remark, p. 11-11 in [12]). One sees easily then that $(e_\varphi)^* = e_{-\varphi}$ or

$$\left(\sum_{\varphi \in \Delta} a_\varphi e_\varphi\right)^* = \sum_{\varphi \in \Delta} \bar{a}_\varphi e_{-\varphi} \qquad (6.1.1)$$

for any set of complex numbers a_φ, $\varphi \in \Delta$.

6.2. We recall now (see §6.1) that $\underline{g}$ is simple. Let $\psi \in \Delta$ be the highest root.[6] Let $\Pi^Q \subseteq \Delta$ be the subset of $\ell + 1$ roots obtained by adjoining $-\psi$ to the simple positive roots. That is, $\Pi^Q = \Pi \cup (-\psi)$. The notion of simple root and highest root are notions which of course are relative to the choice of a lexicographical ordering in $\underline{h}^*$ or rather to the choice of a Weyl chamber (see § 5.1). We will say then that the roots in Π^Q are Q-simple relative to the chamber D. (We shall not require the fact but it can be shown that the number of Weyl chambers which give rise to the same set of Q-simple roots is equal to order of the fundamental group of G; that is, to the order of the center of the simply connected covering group of G.)

Now an element $z \in \underline{g}$ will be called cyclic if there exists a Cartan subalgebra $\underline{h}_1$ and a set $\Pi_1^Q \subseteq \Delta_1$ of Q-simple roots relative to some Weyl chamber D_1 in $\underline{h}_1^*$ such that z can be written

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6. The properties of which will be required are all consequences of (5.3.5).

$$z = \sum_{\xi \in \Pi_1^Q} a_\xi e_\xi \qquad (6.2.1)$$

where the a_ξ, $\xi \in \Pi_1^Q$, are non-zero complex numbers and the e_ξ are root vectors for $\underline{h}_1$ corresponding to the roots $\xi \in \Delta_1$.[7] Cyclic elements play a major role in the remainder of this paper.

If z is given by (6.2.1) observe that in effect we have formed the cyclic element z by adding $a_{-\psi_1} e_{-\psi_1}$ to a principle nilpotent element where ψ_1 is the highest root in Δ_1 relative to D_1. It shall be shown that not only does this destroy nilpotency but that cyclic elements are in fact regular. First we shall need

Lemma 6.2. *Let the cyclic elements* $z, z' \in \underline{g}$ *be given by*

$$z = \sum_{\beta \in \Pi^Q} a_\beta e_\beta$$

and

$$z' = \sum_{\beta \in \Pi^Q} a'_\beta e_\beta$$

where the coefficients a_β, a'_β *are all arbitrary non-zero complex numbers. Then if* H *is the subgroup of* G *corresponding to* $\underline{h}$ *there exists an element* $A \in H$ *and a non-zero scalar* λ *such that*

$$Az = \lambda z'$$

Proof. Let ϵ_i, $i = 1,2,\ldots,\ell$ be as in §5.1. Let $y \in \underline{h}$ be given by

$$y = \sum_{i=1}^{\ell} \mathrm{Log}\left(a'_{\alpha_i} / a_{\alpha_i} \right) \epsilon_i$$

---

7. The set Δ_1 is the set of roots associated with the Cartan subalgebra $\underline{h}_1$.

Clearly Exp y ϵ H and

$$\text{Exp } y \left(\sum_{i=1}^{\ell} a_{\alpha_i} e_{\alpha_i} \right) = \sum_{i=1}^{\ell} a'_{\alpha_i} e_{\alpha_i} \quad .$$

Now let c be any complex number and let x_o be given by (5.2.1). Then clearly

$$\text{Exp } (y + cx_o) \left(\sum_{i=1}^{\ell} a_{\alpha_i} e_{\alpha_i} \right) = e^c \left(\sum_{i=1}^{\ell} a'_{\alpha_i} e_{\alpha_i} \right) . \quad (6.2.1)$$

Now let $q = o(\psi)$ as in §5.3. But then

$$\text{Exp } cx_o(e_{-\psi}) = e^{-cq} e_{-\psi}$$

and if b is defined by Exp $y(e_{-\psi}) = be_{-\psi}$ then $b \neq 0$ and

$$\text{Exp } (y + cx_o)(a_{-\psi} e_{-\psi}) = e^{-cq} b\, a_{-\psi} e_{-\psi} \quad . \quad (6.2.2)$$

Now choose c so that

$$e^{-c(q+1)} = a'_{-\psi} / b a_{-\psi} \quad .$$

Then

$$e^{-cq} b a_{-\psi} = e^c a'_{-\psi} \quad . \quad (6.2.3)$$

Hence if $\lambda = e^c$, $A = \text{Exp}(y + cx_o)$, one has A ϵ H and (6.2.1), (6.2.2), and (6.2.3) imply

$$A \left(\sum_{\beta \epsilon \Pi Q} a_\beta e_\beta \right) = \lambda \left(\sum_{\beta \epsilon \Pi Q} a'_\beta e_\beta \right) \quad . \qquad \text{Q.E.D.}$$

An immediate consequence of Lemma 6.2 is the following theorem which asserts that up to scalar multiplication the set of cyclic elements forms a single conjugate class in g.

Theorem 6.2. *Let* z *and* z' *be any two cyclic elements in* $\underline{g}$. *Then there exists a scalar* λ, $\lambda \neq 0$, *such that* z *and* $\lambda z'$ *are conjugate to each other.*

Proof. If D_1 is a Weyl chamber in $\underline{h}_1^*$ where $\underline{h}_1$ is a Cartan subalgebra there exists $A \in G$ such that $A\underline{h}_1 = \underline{h}_1$ and $AD_1 = D$. This fact together with Lemma 6.2 proves Theorem 6.2. Q.E.D.

6.3. Lemma 6.1 is needed solely to prove the following lemma.

Lemma 6.3. *Cyclic elements are semi-simple.*

Proof. Now if we write

$$\psi = \sum_{i=1}^{\ell} q_i \alpha_i \qquad (6.3.1)$$

recall that the coefficients q_i are positive integers. (See (5.3.3)). Thus if e_1 is defined by

$$e_1 = \sum_{i=1}^{\ell} \sqrt{q_i}\ e_{\alpha_i}$$

and z_1 is defined by

$$z_1 = e_1 + e_{-\psi}$$

then e_1 and z_1 are, respectively, principle nilpotent and cyclic elements.

To prove Lemma 6.3 it suffices by Theorem 6.2 to prove only that z_1 is semi-simple. But to prove that z_1 is semi-simple, by Lemma 6.1 it suffices only to prove that z_1 is normal with respect to $\underline{k}$. Now by (6.1.1)

$$z_1^* = e_1^* + e_\psi$$

and

$$e_1^* = \sum_{i=1}^{\ell} \sqrt{q_i}\ e_{-\alpha_i} \quad .$$

But, obviously then, since ψ is the highest root

$$[e_1, e_\psi] = [e_1^*, e_{-\psi}] = 0 \quad .$$

Thus

$$[z_1, z_1^*] = [e_1 + e_{-\psi}, e_1^* + e_\psi] = [e_1, e_1^+] + [e_{-\psi}, e_\psi].$$

But

$$[e_1, e_1^*] = \sum_{i=1}^{\ell} q_i [e_{\alpha_i}, e_{-\alpha_i}] = \sum_{i=1}^{\ell} q_i \alpha_i = \psi$$

by (5.1.1) and (6.3.1). On the other hand, of course,

$$[e_{-\psi}, e_\psi] = -\psi \quad .$$

Thus $[z_1, z_1^*] = 0$ and hence z_1 is normal with respect to $\underline{k}$.

Q.E.D.

6.4. To gain information on cyclic elements in general it suffices by Theorem 6.2 to focus attention on a single fixed cyclic element. We choose this element to be z_o where z_o is given by

$$z_o = e_o + e_{-\psi} \quad . \tag{6.4.1}$$

Here, of course, e_o is given by (5.2.2).

As in § 5.2 let $\underline{a}_o$ be the TDS spanned by x_o, e_o and f_o. Now let n_k be the multiplicity of the irreducible $2k+1$-dimensional representation π_{2k+1} in the complete reduction of the

adjoint representation of $\underline{a}_0$ on $\underline{g}$. By Corollary 5.2, $n_k = 0$ if k is a half-integer and

$$\ell = n_0 + n_1 + n_2 + \dots \qquad (6.4.2)$$

Now it follows from §2.5(d) and (5.2.4) that the kernel $\underline{g}^{e_0}$ of ad e_0 is contained in $\underline{s}$. On the other hand it is obvious that $\underline{g}^{e_0} \cap \underline{h} = 0$. Therefore since $x_0 \in \underline{h}$ is regular it follows that the eigenvalue of ad x_0 on $\underline{g}^{e_0}$ are positive. This implies two things, (1) $\underline{g}^{e_0} \subseteq \underline{n}$ (Actually we have already noted this fact - see proof of Corollary 5.6) and (2) by §2.5(d), $n_0 = 0$. That is, only the zero element is annihilated by ad $\underline{a}_0$. Indeed this also follows from §2.5(h) which asserts

$$\begin{aligned} n_0 &= \dim \underline{g}_0 - \dim \underline{g}_1 \\ &= \ell - \ell \\ &= 0 \end{aligned}$$

For the present we direct our attention to the first fact, $\underline{g}^{e_0} \subseteq \underline{n}$.

Let $\underline{n}^*$ be the maximal Lie subalgebra of nilpotent elements given by

$$n^* = \sum_{\varphi \in \Delta^+} (e_{-\varphi}) \quad .$$

Then, of course,

$$\underline{g} = \underline{n}^* + \underline{h} + \underline{n} \qquad (6.4.3)$$

is a direct sum decomposition. We shall let ρ (resp. ρ^*) be the projection

$$\rho : \underline{g} \longrightarrow \underline{n}$$

$$(\text{resp. } \rho^* : \underline{g} \longrightarrow \underline{n}^*)$$

of $\underline{g}$ onto $\underline{n}$ (resp $\underline{n}^*$) defined by the decomposition (6.4.3).

Now consider the kernel $\underline{g}^{z_o}$ of ad z_o. A relation between $\underline{g}^{e_o}$ and $\underline{g}^{z_o}$ is given by

Lemma 6.4A. *Let ρ_o be the restriction of the projection to the subspace $\underline{g}^{z_o}$. Then $\rho_o(\underline{g}^{z_o}) \subseteq \underline{g}^{e_o}$ and in fact*

$$\rho_o : \underline{g}^{z_o} \longrightarrow \underline{g}^{e_o}$$

is a linear isomorphism of $\underline{g}^{z_o}$ onto $\underline{g}^{e_o}$.

Proof. It follows immediately from the decomposition (6.4.3) of $\underline{g}$ and (5.1.3) that

$$[e_{-\psi}, \underline{n}] \subseteq \underline{n}^* + \underline{h}$$
$$[e_{-\psi}, \underline{h}] \subseteq (e_{-\psi})$$
$$[e_{-\psi}, \underline{n}^*] = 0 \quad .$$

Now let $y \in \underline{g}^{z_o}$ be arbitrary. Write

$$y = v + x + u$$

where $v \in \underline{n}^*$, $x \in \underline{h}$, and $u \in \underline{n}$.

Now by (6.4.1) and §5.1

$$\begin{aligned} 0 &= [z_o, y] \\ &= [e_o, v] + [e_o, x] + [e_o, u] \\ &\quad + (x, \psi) e_{-\psi} + [e_{-\psi}, u] \end{aligned} \tag{6.4.4}$$

Applying the projection ρ to both sides of (6.4.4) we obtain

$$[e_o, x] + [e_o, u] = 0 \ .$$

But then recalling the relation $[\underline{g}_i, \underline{g}_j] \subseteq \underline{g}_{i+j}$, it is

clear that $[e_o,x] = 0$ and $[e_o,u] = 0$. Thus since $u = \rho(y)$ it follows that $\rho(\underline{g}^{z_o}) \subseteq \underline{g}^{e_o}$. It also follows that since $\underline{g}^{e_o} \subseteq \underline{n}$ we must have

$$x = 0. \tag{6.4.5}$$

Thus $y = v+u$ and

$$0 = [e_o,v] + [e_o,u] + [e_{-\psi},u] \quad .$$

To show first that ρ_o is one-one, assume $u = 0$. But this and the last expression imply that $v \in \underline{g}^{e_o}$. But $\underline{g}^{e_o} \subseteq \underline{n}$. Hence $v = 0$. Thus $y = 0$ which proves that ρ_o is an isomorphism into. But now, by Corollary 5.3, $\dim \underline{g}^{e_o} = \ell$. On the other hand $\dim \underline{g}^{z_o} \geq \ell$ (see, for example Theorem 5.7). Since ρ_o is an isomorphism this means

$$\dim \underline{g}^{z_o} = \ell$$

and also that ρ_o is onto. Q.E.D.

A major consequence of Lemma 6.4A is

Corollary 6.4. Cyclic elements are regular.

Proof. As above let z_o be defined by (6.4.1). It suffices to show z_o is regular. But by Lemma 6.3 z_o is semi-simple. Hence $\underline{g}^{z_o}$ contains a Cartan subalgebra. But, by Lemma 6.4, $\dim \underline{g}^{z_o} = \ell$. Hence z_o is regular. Q.E.D.

Corollary 6.4 implies

Lemma 6.4B. Let z_o be defined by (6.4.1). Then $\underline{g}^{z_o}$ is a Cartan subalgebra.

We will let $\underline{h}'$ designate the Cartan subalgebra $\underline{g}^{z_o}$.

In the proof of Lemma 6.4 we considered the decomposition of any element $y \in \underline{h}'$ and showed that its projection x in $\underline{h}$ vanishes. (See (6.4.5)). We state this as

Lemma 6.4C. *Let z_o be defined by* (6.4.1). *Let ρ and ρ_* be the projections of g on n and n^* also defined as in* §6.4. *Then the projection $1-(\rho+\rho^*)$ of g on $\underline{h}$ vanishes on $\underline{h}'$.*

Perhaps a simpler way of expressing Lemma 6.4C is to state that the Cartan subalgebras $\underline{h}'$ and $\underline{h}$ are orthogonal to each other (with respect to B).

6.5. Now g^{e_o} is invariant under ad x_o (see §2.5(d)). Let u_i, $i = 1,2,\ldots,\ell$ be a basis of g^{e_o} and also eigenvectors of ad x_o. Let k_i, $i = 1,2,\ldots,\ell$ be the corresponding eigenvalues. That is, $u_i \in g_{k_i}$, $i = 1,2,\ldots,\ell$. We may regard the basis so ordered that $k_i \leq k_{i+1}$. At this stage we already know two of the k_i, namely, the extreme ones k_1 and k_ℓ. We also know an inequality, $k_{\ell-1} < k_\ell$.

Lemma 6.5A. *Let k_i be defined as above. Then where $q = o(\psi)$*

$$1 = k_1 \leq k_i \leq \cdots \leq k_{\ell-1} < k_\ell = q .$$

Proof. As we noted in §6.4 we must have $k_1 > 0$. On the other hand since $e_o \in g^{e_o}$ it follows that $k_1 = 1$. But also $e_\psi \in g^{e_o}$ since ψ is the highest root. Thus $[x_o, e_\psi] = qe_\psi$ implies $k_\ell = q$. Since $o(\varphi) < q$ for every $\varphi \neq \psi$ (see (5.3.4)) it also follows that $k_{\ell-1} < k_\ell$. Q.E.D.

Taking the proof of Lemma 6.5A into account we will choose $u_1 = e_o$ and $u_\ell = e_\psi$.

Now let d_i, $i = 1,2,\ldots,\ell$ be the dimensions, in non-decreasing order, of the irreducible components occurring in the complete reduction of adjoint representation of a principle TDS (e.g. $\underline{a}_o$) on $\underline{g}$. Applying §2.5(d) the eigenvalues k_i yield the dimensions d_i by the relation

$$d_i = 2k_i+1 . \tag{6.5.1}$$

Lemma 6.5A implies $d_1 = 3$, $d_\ell = 2q+1$ and $d_{\ell-1} < 2q+1$.

Now by Lemma 6.4A there exists a unique basis $y_1, y_2, \ldots, y_\ell$ of the Cartan subalgebra $\underline{h}'$ with the property that $\rho(y_i) = u_i$. Define v_i by

$$y_i = u_i + v_i . \tag{6.5.2}$$

By Lemma 6.4C it follows that $v_i \in \underline{n}^*$ for $i = 1,2,\ldots,\ell$. Since $u_1 = e_o$ and since $\rho(z_o) = e_o$ it follows from (6.4.1) and Lemma 6.4A that $v_1 = e_{-\psi}$ and $y_1 = z_o$. Now $u_\ell = e_\psi$. We shall need to know what v_ℓ is.

Lemma 65.B. *Let y_i, u_i and v_i, $i = 1,2,\ldots,\ell$ be defined as above. Let the non-zero scalars c_i and q_i, $i = 1,2,\ldots,\ell$ be defined respectively by* (5.2.2) *and* (6.3.1). *Then v_ℓ is the element given by*

$$v_\ell = \sum_{i=1}^{\ell} q_i/c_i \; e_{-\alpha_i} . \tag{6.5.3}$$

Proof. As in the proof of Lemma 6.3

$$\begin{aligned} &[z_o, e_\psi + \sum_{i=1}^{\ell} q_i/c_i \; e_{-\alpha_i}] \\ &\quad = [e_{-\psi}, e_\psi] + \sum_{i=1}^{\ell} q_i [e_{\alpha_i}, e_{-\alpha_i}] \\ &\quad = -\psi + \sum_{i=1}^{\ell} q_i \alpha_i = 0 \end{aligned}$$

Thus $e_{\psi} + \sum_{i=1}^{\ell} q_i/c_i \; e_{-\alpha_i} \in \underline{h}'$. It follows immediately then from Lemma 6.4A that $v_{\ell} = \sum_{i=1}^{\ell} q_i/c_i \; e_{-\alpha_i}$. Q.E.D.

6.6. Now it is clear (for example from the matrix representation of a TDS given in §2.5) that there exists an element A in the subgroup of G corresponding to $\underline{a}_o$ such that $Ax_o = -x_o$. But since x_o is regular it is clear then that

$$A : \underline{h} \longrightarrow \underline{h}$$
$$A : \underline{n}^* \longrightarrow \underline{n}$$

In fact these relations are already contained in the more general fact

$$A : \underline{g}_j \longrightarrow \underline{g}_{-j}$$

for any $-q \leq j \leq q$. It follows then that we may interchange the roles of Δ^+ and $-\Delta^+$, $\underline{n}$ and $\underline{n}^*$ and algo $\underline{g}_1$ and $\underline{g}_{-1}$ in the results of §§ 6.2-5. But then one sees that v_{ℓ} is principle nilpotent and that $y_{\ell} = u_{\ell} + v_{\ell}$ is cyclic. But then by Corollary 6.4 it follows that $g^{y_{\ell}} = \underline{h};$. Finally applying Lemma 6.4A we obtain

Lemma 6.6. *We retain previous notation. Let ρ_o^* be the restriction of ρ^* to $\underline{h}'$. Then (1) v_{ℓ} is a principle nilpotent element, (2) $\rho_o^*(\underline{h}') \subseteq \underline{g}^{v_{\ell}}$ and in fact*

$$\rho_o^* : \underline{h}' \longrightarrow \underline{g}^{v_{\ell}}$$

is isomorphism onto. In other words the elements v_i, $i = 1,2,\ldots\ell$ form a basis of $\underline{g}^{v_{\ell}}$. Also the elements v_i, commute with each other. (See Corollary 5.8).

It is obvious from (5.2.3) that v_ℓ is a nil-negative element of a principle S-triple containing x_0 as neutral element or an nil-positive element of a principle S-triple containing $-x_0$ as neutral element. Thus g^{v_ℓ} is stable under ad x_0 and since all principle S-triples are conjugate the eigenvalues of ad x_0 on g^{v_ℓ} are $-k_\ell$, $-k_{\ell-1},\ldots,-k_1$ in non-decreasing order. The elements v_i are a basis of g^{v_ℓ} by Lemma 6.6 but it is not at all clear yet that they are eigenvectors of ad x_0.

6.7. We now isolate a particular conjugate class in G, elements of which, will play a major role in the remainder of this paper. An element $P \in G$ will be called a principle element of G if there exists a principle regular element $x \in \underline{g}$ (see §5.2) such that P can be written

$$P = \operatorname{Exp}\left(\frac{2\pi i}{s}\right) x \qquad (6.7.1)$$

where $s = q+1$ and q, as usual, is the order of the highest root ψ. Note that since x lies in a TDS, the principle element P lies in a subgroup of G corresponding to a TDS. (We shall make no use of the fact here but it can be shown that the number of principle regular elements x which satisfy (6.7.1) for a fixed principle element $P \in G$ is equal to order of the fundamental group of G). Various characterizations of principle elements will be given in §§8 and 9.

Throughout the remainder of the paper we will let ω be the primitive s root of unity defined by $\omega = e^{\frac{2\pi i}{s}}$. Let P_0 be principle element in G defined by letting $P_0 = \exp\left(\frac{2\pi i}{s}\right) x_0$. It

is clear then that

$$P_o u = \omega^j u$$

for $u \in \underline{g}_j$, $-q \leq j \leq q$. It follows then that ω^j, $j=0,1,\ldots,q$ are the eigenvalues of G and if $\underline{u}_j$ designates the corresponding eigenspaces then

$$\underline{u}_j = \underline{g}_j + \underline{g}_{j-s} \tag{6.7.2}$$

where it is understood that $\underline{g}_{-s}$ denotes the zero subspace.

It is an immediate consequence of (6.7.2) that z_o is an eigenvector of P_o. In fact

$$P_o z_o = \omega z_o \tag{6.7.3}$$

But this clearly means that the Cartan subalgebra $\underline{h}' = \underline{g}^{z_o}$ is stable under P_o. On the other hand the elements $y_i = u_i + v_i$ where $u_i \in \underline{g}_{k_i}$ form a basis of $\underline{h}'$. Let us apply P_o to these basal elements. Then by §6.5

$$P_o y_i = \omega^{k_i} u_i + P_o v_i$$

But the elements $P_o y_i$ belong to $\underline{h}'$ and furthermore since, obviously, $P_o v_i \in \underline{n}^*$ it follows that

$$\rho_o(P_o y_i) = \omega^{k_i} u_i \quad .$$

But now if we apply Lemma 6.4A we must have

$$P_o y_i = \omega^{k_i} y_i \quad .$$

Thus y_i, u_i and hence also v_i are contained in $\underline{u}_{k_i}$. But since $v_i \in \underline{n}^*$ it follows from (6.7.2) that $v_i \in \underline{g}_{k_i - s}$. Thus not only did we prove that the v_i are eigenvectors of ad x_o but more we obtain a duality relation among the integers k_i.

Lemma 6.7. Let v_i, $i = 1,2,\ldots, \ell$ be defined by (6.5.2). Then $v_i \in g_{k_i - s}$, $i = 1,2,\ldots, \ell$. Also the integers k_i satisfy the following duality law,

$$s = k_1 + k_\ell = k_2 + k_{\ell-1} = \cdots = k_\ell + k_1 .$$

Proof. Only the second part of Lemma 6.7 is not yet proved. By Lemma 6.6 the elements v_i are a basis of g^{v_ℓ} and the first part of Lemma 6.7 asserts that v_i is an eigenvector of ad x_0 with eigenvalue $k_i - s$. On the other hand as we have seen the eigenvalues of ad x_0 on g^{v_ℓ} are, in non-decreasing order,

$$-k_\ell \leq -k_{\ell-1} \leq \cdots \leq -k_1 .$$

But these must be identical, and in the same order, as

$$k_1 - s \leq k_2 - s \leq \quad \leq k_\ell - s .$$

Subtracting termwise we obtain for all defined p, $k_p + k_{\ell+1-p} = s$.

Q.E.D.

Observe that the duality together with the inequality $k_{\ell-1} < k_\ell$ implies the inequality $k_1 < k_2$.

In [3] Chevalley observed empirically a similar duality in the exponents (see [3], p. 24) of g. Later it will be proved that the integers k_i are in fact these exponents and that the duality observed in the latter is the same as the duality just proved in the former. More than just a numerical coincidence the duality in the k_i takes the following form. (Theorem 6.7 summarizes results which have already been proved.)

Theorem 6.7. Let c_i, $i = 1,2,\ldots, \ell$ be ℓ arbitrary

non-zero complex numbers. Let q_i, $i = 1,2,\ldots,\ell$ be the coefficients of the highest root ψ relative to the simple positive roots α_i.

Let

$$e_o = \sum_{i=1}^{\ell} c_i e_{\alpha_i}$$

and

$$\tilde{e}_o = \sum_{i=1}^{\ell} q_i / c_i \; e_{-\alpha_i} \quad .$$

Then e_o and $\tilde{e}_o$ are principle nilpotent elements in g so that g^{e_o} and $g^{\tilde{e}_o}$ are ℓ-dimensional and commutative.

Let $x_o = \sum_{i=1}^{\ell} \epsilon_i$, where ϵ_i, $i = 1,2,\ldots,\ell$ be the basis of the Cartan subalgebra h dual to the simple positive roots α_i, $i = 1,2,\ldots,\ell$, so that $(x_o,\varphi) = o(\varphi)$ is the order of a root φ. For any non-zero integer let

$$g_j = \sum_{o(\varphi)=j} (e_\varphi)$$

be the eigenspace of ad x_o for the eigenvalue j.

Then g^{e_o} and $g^{\tilde{e}_o}$ are each stable under ad x_o and the eigenvalues of the restriction of ad x_o to g^{e_o} and $g^{\tilde{e}_o}$, respectively, are positive integers k_i and the negative integers $-k_i$, $i = 1,2,\ldots,\ell$ where

$$1 = k_1 < k_2 \leq \ldots \leq k_{\ell-1} < k_\ell = q,$$

and $q = \sum_{i=1}^{\ell} q_i = o(\psi)$ and $2k_1+1$, $2k_2+1,\ldots,2k_\ell+1$ are the dimensions of the irreducible components occurring in the complete reduction of the adjoint representation of a principle TDS on g.

Let

$$z_o = e_o + e_{-\psi}$$

and

$$\tilde{z}_o = \tilde{e}_o + e_{\psi}$$

(cyclic elements). Then z_o and $\tilde{z}_o$ are regular, so that g^{z_o} and $g^{\tilde{z}_o}$ are Cartan subalgebras; and $[z_o, \tilde{z}_o] = 0$ so that $g^{z_o} = g^{\tilde{z}_o}$.

Let P_o, a principle element of G, be given by $P_o = \mathrm{Exp}\left(\frac{2\pi i}{s}\right)x_o$ where $s = q+1$. Write h' for $g^{z_o} = g^{\tilde{z}_o}$. Then h' is orthogonal to h. But much more than this, h' is stable under P_o and the eigenvalues of $P_o|h'$ are ω^{k_i}, $i = 1,2,\ldots,\ell$ where $\omega = e^{\frac{2\pi i}{s}}$ Furthermore we can find corresponding eigenvectors, y_i, which form a basis of h' and are such that

$$y_i = u_i + v_i$$

where the u_i form a basis of g^{e_o} and the v_i form a basis of $g^{\tilde{e}_o}$, $i = 1,2,\ldots,\ell$. Moreover, $u_i \in g_{k_i}$ and $v_i \in g_{k_i - s}$ so that the integers k_i satisfy the duality relation

$$s = k_1 + k_\ell = k_2 + k_{\ell-1} = \cdots = k_\ell + k_1 .$$

Finally we can choose $y_1 = z_o$ and $y_\ell = \tilde{z}_o$.

6.8. Among other things, Theorem 6.7 asserts that the integer k_i may be found by considering the eigenvalues, ω^{k_i}, of the restriction of the principle element P_o to h'. Theorem 6.8 below asserts that without knowledge of h' or the restriction $P_o|h'$ we can determine the integers k_i directly by considering

multiplicities of eigenvalues of a principle element with respect to its action on $\underline{g}$. In the notation of §6.4 and Theorem 6.8 this means we can express n_k in terms of s_k.

Theorem 6.8. *Let P be a principle element of G. Let s_k, $k = 0,1,\ldots,q$ be the multiplicity of the eigenvalue ω^k of P. (in the notation of §6.7, $s_k = \dim \underline{g}_k + \underline{g}_{k-s}$) and as in §6.4 let n_k be the multiplicity of π_{2k+1} in the adjoint representation a principle TDS on $\underline{g}$. Then*

$$n_k = s_k - \ell \tag{6.8.1}$$

In other words let $\underline{h}_1$ be a Cartan subalgebra which is stable under P and is such that the eigenvalues of $P|\underline{h}_1$ are ω^{k_i}, $i = 1,2,\ldots,\ell$ (such a Cartan subalgebra exists by Theorem 6.7). Then if $\underline{m}_1$ is the B-orthocomplement to $\underline{h}_1$, $\underline{m}_1$ is stable under P and the eigenvalues of $P|\underline{m}_1$ are ω^k, $k = 0,1,\ldots,q$. Furthermore each eigenvalue of $P|\underline{m}_1$ occurs with multiplicity ℓ.

Proof. Since any element in G is orthogonal with respect to B it follows that $\underline{m}_1$ is stable under P. It suffices then to assume $P = P_o$ and $\underline{h}_1 = \underline{h}$ to prove Theorem 6.8 for this case. Now by §2.5(h)

$$\dim \underline{g}_j - \dim \underline{g}_{j+1} = n_j . \tag{6.8.2}$$

Since $n_0 = 0$ upon summing (6.8.2) we obtain for $0 \leq k \leq q$

$$\dim \underline{g}_0 - \dim \underline{g}_k = \sum_{j=1}^{k-1} n_j . \tag{6.8.3}$$

But by §2.5(f)

$$\dim \underline{g}_{s-k} = \sum_{j=s-k}^{q} n_j . \tag{6.8.4}$$

But by the duality in Theorem 6.7 $n_{s-j} = n_j$. But then subtracting (6.8.3) from (6.8.4) and recalling that dim g_{s-k} = dim g_{k-s} and dim $g_0 = \ell$ it follows that

$$s_k - \ell = \dim(g_k + g_{k-s}) - \ell = n_k . \qquad \text{Q.E.D.}$$

Corollary 6.8. Let r be the number of positive roots of g. Let s = q+1 where q is the order of the highest root ψ. Then

$$\ell s = 2r .$$

Proof. This immediately follows from the second part of Theorem 6.8 since dim $m_1 = 2r$. Q.E.D.

Since $n = \ell + 2r$ note that it also proved that ℓ divides n. In fact $n = \ell(s+1)$.

7. The notion of the apposition of two Cartan subalgebras. 1. Let h_1 be any Cartan subalgebra of g. Let H_1 be the corresponding subgroup of G. Let $N(H_1)$ be the normalizer of H_1 in G. It is obvious that $A \in N(H_1)$ if and only if h_1 is stable under A. Let W_1 be the subgroup of endomorphisms of h_1 induced by restricting the elements of $N(H_1)$ to h_1. Then the restriction defines a homomorphism

$$\zeta : N(H_1) \longrightarrow W_1$$

of $N(H_1)$ onto W_1. The group W_1 is finite and is called the Weyl group with respect to h_1. It is well known that the kernel of ζ is H_1 so that $N(H_1)/H_1 \cong W_1$. If $\sigma \in W_1$ and $A \in N(H_1)$ is such that $\zeta(A) = \sigma$ then A will be called an extension of σ. It is clear then that the most general extension of σ is of the

form $A_1 A$ where $A_1 \in H_1$.

Let Δ_1 be the set of roots with respect to $\underline{h}_1$. It is well known that Δ_1 is stable under W_1. For each root $\varphi \in \Delta_1$ let R_φ be the "reflection" of $\underline{h}_1$ defined by

$$R_\varphi y = y - \frac{2(\varphi, y)}{(\varphi, \varphi)} \varphi \qquad (7.1.1)$$

One knows that $R_\varphi \in W_1$ for all $\varphi \in \Delta_1$ and that W_1 is generated by these reflections (See [12], Theorème 1, p. 16-05).

Lemma 7.1. Let $\sigma \in W_1$. Let μ be an eigenvalue of σ. Then μ is a primitive j^{th} root of unity for some j and every other primitive j^{th} root of unity occurs as an eigenvalue of σ.

Proof. Since Δ_1 is stable under σ there exists a basis of $\underline{h}_1$ (e.g. a set of simple positive roots) with respect to which σ is represented by a matrix with rational coefficients. It follows then that the characteristic polynomial of σ also has rational coefficients. Now since σ has finite order μ must be a primitive j^{th} root of unity for some integer j. It follows then that the characteristic polynomial of σ is a product of cyclotomic polynomials and hence every primitive j^{th} root of unity is a root of the characteristic polynomial. Q.E.D.

7.2. Let $\underline{u}_1$ and Π^Q, the set of Q-simple roots with respect to D, be defined, respectively, as in §6.7 and §6.2. It is clear that $\underline{u}_1$ is the set of all elements $y \in \underline{g}$ of the form

$$y = \sum_{\beta \in \Pi^Q} a_\beta e_\beta \quad .$$

We know that when all the coefficients a_β are different from zero y is cyclic, and hence also regular. It is interesting that in case one of the coefficients is zero not only does y cease to be regular it is in fact nilpotent.

Lemma 7.2. *Let* Π^Q *be as in* §6.2. *Let* $y \in \underline{g}$ *be of the form*

$$y = \sum_{\beta \in \Pi^Q} a_\beta e_\beta \quad .$$

That is, in the notation of §6.7, *assume* y *satisfies* $P_o y = \omega y$. *Then if one of the coefficients* a_β *is zero,* y *is nilpotent.*

Proof. Assume one of the coefficients is zero. In fact let β_i, $i = 1,2,\ldots, \ell+1$ be the elements of Π^Q. Without loss assume $a_{\beta_{\ell+1}} = 0$.

Now upon writing $\psi = \sum_{i=1}^{\ell} q_i \alpha_i$ we know that the coefficients q_i are all non-zero (See (5.3.3)). It follows immediately then that the roots $\beta_1, \beta_2, \ldots, \beta_\ell$ form a basis of $\underline{h}$. But if we use this basis to define a lexicographical ordering in $\underline{h}^*$ and set $\tilde{\Delta}^+$ equal to the set of positive roots relative to this ordering it follows that $\beta_i \in \tilde{\Delta}^+$ for $i = 1,2,\ldots, \ell$. But then $\tilde{\underline{n}} = \sum_{\varphi \in \tilde{\Delta}^+} (e_\varphi)$ is a Lie algebra of nilpotent elements and $y \in \tilde{\underline{n}}$. Hence y is nilpotent. Q.E.D.

7.3. Applying Lemma 6.2 we obtain the corollary

Lemma 7.3. *Let* $y \in \underline{u}_1$. *Let* H *be the subgroup of* G *corresponding to* $\underline{h}$. *Then* y *is either nilpotent or else there exists* $A \in H$ *and a non-zero scalar* λ *such that*

$$y = \lambda A z_o \quad .$$

For any element $A \in G$ let $\underline{g}^A$ be the set of fixed vectors of A. An element $A \in G$ is then said to be regular if (1) A is semi-simple (that is, A is diagonalizable) and (2) $\underline{g}^A$ is a Cartan subalgebra. It follows from Theorem 6.8 that a principle element $P \in G$ is regular.

Now a special relation, to be defined below, exists between the pair of orthogonal Cartan subalgebras $\underline{h}$ and $\underline{h}'$ (See Theorem 6.7). We will show that this relation defines $\underline{h}$ and $\underline{h}'$ uniquely up to conjugacy.

Let $\underline{h}_1$ and $\underline{h}_1'$ be two Cartan subalgebras of $\underline{g}$. We will then say that $\underline{h}_1'$ is in apposition to $\underline{h}_1$ with respect to a principle element $P \in G$ if (1) $\underline{g}^P = \underline{h}_1$ and, (2) $\underline{h}_1'$ is stable under P and the set of eigenvalues of $P|\underline{h}_1'$ includes a primitive s root of unity. (Observe that the number s has special significance for a principle element; the order of a principle element in G equals s.)

By Theorem 6.7 it is clear that $\underline{h}'$ is in apposition to $\underline{h}$ with respect to P_o.

Theorem 7.3. *Assume that the Cartan subalgebra $\underline{h}_1'$ is in apposition to $\underline{h}_1$ with respect to the principle element P_1 and that $\underline{h}_2'$ is in apposition to $\underline{h}_2$ with respect to P_2. Then there exists $A \in G$ such that $AP_1A^{-1} = P_2$, $A\underline{h}_1 = \underline{h}_2$, and $A\underline{h}_1' = \underline{h}_2'$.*

Proof. It suffices to assume $P_1 = P_o$, $\underline{h}_1 = \underline{h}$ and $\underline{h}_1' = \underline{h}'$. Now all principle elements in G are conjugate to each other. Hence there exists $A_1 \in G$ such that $A_1P_oA_1^{-1} = P_2$. Since

$\underline{g}^{P_2} = \underline{h}_2$ it is obvious that $A_1\underline{h} = \underline{h}_2$. Thus it also suffices to assume $\underline{h}_2 = \underline{h}$ and $P_2 = P_o$. But now $\underline{h}_2'$ is stable under P_o and $P_o|\underline{h}_2'$ has a primitive s root of unity as an eigenvalue. By Lemma 7.1 ω is also an eigenvalue of $P_o|\underline{h}_2'$. Let $y \in h_2'$, $y \neq 0$, be a corresponding eigenvector. Then in the notation of § 6.7, $y \in \underline{u}_1$. But since y is contained in a Cartan subalgebra, y is not nilpotent. Hence by Lemma 7.3 there exists $A \in G$ and a non-zero scalar λ such that $y = \lambda Az_o$. But then by Corollary 6.4 y is regular and hence $A\underline{h}' = \underline{h}_2'$. Q.E.D.

8. The Coxeter-Killing transformation. 1. An element $\gamma' \in W$ will be called a transformation of Coxeter-Killing if it can be put in the form

$$\gamma' = R_{\alpha_1'} R_{\alpha_2'} \cdots R_{\alpha_\ell'}$$

where α_i' , $i = 1,2,\ldots,\ell$ are the simple positive roots, in some order, relative to some Weyl chamber in $\underline{h}^{\#}$. In recent years this transformation has been studied by Coxeter. Relationships between the eigenvalues of this transformation and the eigenvalues of the matrix of Cartan integers have been established. (See [5]). Also one knows that all the transformations of Coxeter-Killing in W form a conjugate class in W.

Let $\gamma = R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_\ell}$. An integral valued function on the Weyl group, which plays an important role in the cohomology theory of complex homogeneous spaces and also of $\underline{n}$ is the function $N(\sigma)$ which assigns to each $\sigma \in W$ the number of positive roots

which go over into negative roots under the action of σ . The knowledge of $N(\gamma)$ will play an essential role in proving that the order of γ is s.

Theorem 8.1. Let γ be the transformation of Coxeter-Killing defined by letting $\gamma = R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_\ell}$. Then $N(\gamma) = \ell$. Furthermore if φ_i, $i = 1,2,\ldots,\ell$ are the positive roots which change sign under γ then the roots φ_i form a basis of $\underline{h}$.

Proof. Let φ_i, $i = 1,2,\ldots,\ell$ be defined by

$$\varphi_i = R_{\alpha_\ell} R_{\alpha_{\ell-1}} \cdots R_{\alpha_{i+1}} \alpha_i \quad .$$

It follows easily from (7.1.1) that

$$\varphi_i = \alpha_i + \sum_{j>i} c_{ij}\alpha_j \quad .$$

Since the coefficient of α_i in φ_i is one it follows that φ_i is positive. Furthermore since the coefficient of α_j in φ_i for $j < i$ is zero it is clear that φ_i are linearly independent and hence form a basis of $\underline{h}$.

But clearly

$$\gamma(\varphi_i) = R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_{i-1}}(-\alpha_i)$$

and hence

$$\gamma(\varphi_i) = -\alpha_i + \sum_{j<i} c'_{ij}\alpha_j \quad .$$

Since the coefficient of α_i in $\gamma(\varphi_i)$ is minus one φ_i is a positive root which changes sign under γ . Thus $N(\gamma) \geq \ell$.

Now it is well known that $N(R_{\alpha_i}) = 1$. In fact α_i is the only positive root which changes sign under R_{α_i}. This is clear

since only the coefficient of α_i in φ, for any $\varphi \in \Delta$, is effected by R_{α_i}.

Now assume φ is a positive root which changes sign under γ. Since $\gamma(\varphi) \in \Delta^-$ there clearly exists a maximal value i such that

$$R_{\alpha_i} R_{\alpha_{i+1}} \cdots R_{\alpha_\ell} \varphi \in \Delta^- .$$

That is, $R_{\alpha_{i+1}} \cdots R_{\alpha_\ell} \varphi \in \Delta^+$. It follows then that

$$\alpha_i = R_{\alpha_{i+1}} \cdots R_{\alpha_\ell} \varphi .$$

This, however, means that $\varphi = \varphi_i$. Thus $N(\gamma) = \ell$. Q.E.D.

Now it is well known that one is not an eigenvalue of a Coxeter-Killing transformation. A proof of this resting on the work of Coxeter is given, for example, in [4]. (See [4], p. 352). However, since a direct proof of this fact, making use of Theorem 8.1, can be given we shall include it.

Lemma 8.1. Let γ be the Coxeter-Killing transformation given as in §8.1. The one is not an eigenvalue of γ.

Proof. Let ϵ_i, $i = 1,2,\ldots,\ell$ be the basis of $\underline{h}$ given as in §5.1. Let $\epsilon_i' = \frac{2\epsilon_i}{(\alpha_i,\alpha_i)}$. Clearly

$$\begin{aligned} \gamma \epsilon_i' &= R_{\alpha_\ell} \cdots R_{\alpha_i} \epsilon_i' \\ &= \epsilon_i' + R_{\alpha_\ell} \cdots R_{\alpha_{\ell+1}}(-\alpha_i) \end{aligned}$$

by (7.1.1). On the other hand it is clear from the definition of φ_i in the proof of Theorem 8.1 that $R_{\alpha_\ell} \cdots R_{\alpha_{i+1}}(-\alpha_i) = -\gamma(\varphi_i)$. Thus we have

$$\gamma\epsilon_i' = \epsilon_i' + \gamma(\varphi_i) .$$

Assume now that $x \in \underline{h}$ is fixed under γ. Write $x = \sum_{i=1}^{\ell} a_i \epsilon_i'$. Then

$$\begin{aligned} x &= \gamma x \\ &= \sum_{i=1}^{\ell} a_i(\epsilon_i' + \gamma(\varphi_i)) \\ &= x + \sum_{i=1}^{\ell} a_i \gamma(\varphi_i) . \end{aligned}$$

Thus $\sum_{i=1}^{\ell} a_i \gamma(\varphi_i) = 0$. However, by Theorem 8.1 the vectors φ_i and hence $\gamma(\varphi_i)$ are linearly independent. Thus $a_i = 0$, $i = 1,2,\ldots,\ell$ and hence $x = 0$. Q.E.D.

8.2. Lemma 8.1 is needed solely to prove

Lemma 8.2. *Let* $\Gamma_i \subseteq \Delta$, $i = 1,2,\ldots, L$ *be the orbits in* Δ *under the action of* γ. *Then for* $i = 1,2,\ldots,L$

$$\sum_{\varphi \in \Gamma_i} \varphi = 0.$$

Proof. This follows immediately from Lemma 8.1 since the sum of the roots in any orbit of γ is obviously left fixed by γ. Q.E.D.

The following theorem is proved in [4]

Theorem 8.2 (Coleman). *Let* h *be the order of the Coxeter-Killing transformation* γ *defined as in* §8.1. *Let* $\nu = e^{\frac{2\pi i}{h}}$. *Then there exists a regular eigenvector* z_1 *of* γ *whose corresponding eigenvalue is* ν.

Coleman also observed and used in [4] the following consequence of Theorem 8.2. We repeat his proof.

Corollary 8.2. As in Lemma 8.2 let $\Gamma_i \subseteq \Delta$, $i = 1,2,\ldots,L$ denote the distinct orbits of γ with respect to its action on the set of roots Δ. Then each orbit Γ_i contains exactly h roots so that in particular $hL = 2r$ where $2r$ is the total number of roots.

Proof. Let $\varphi \in \Delta$. Assume $\gamma^m \varphi = \varphi$. It suffices to prove that h divides m. Now where z_1 is given as in Theorem 8.2

$$\begin{aligned}(z_1, \varphi) &= (z_1, \gamma^m \varphi) \\ &= (\gamma^{-m} z_1, \varphi) \\ &= \nu^{-m}(z_1, \varphi) \quad .\end{aligned}$$

But since z_1 is regular $(z_1, \varphi) \neq 0$ and hence $\nu^{-m} = 1$. This implies h divides m. Q.E.D.

8.3. Now one knows that the Poincaré polynomial $P_G(t)$ can be put in the form

$$P_G(t) = (1 + t^{2m_1+1})(1 + t^{2m_2+1}) \ldots (1 + t^{2m_\ell+1})$$

where the m_i, $i = 1,2,\ldots,\ell$ are positive integers in non-decreasing order. The integers m_i (sometimes m_i+1) are called the exponents of g (or W as in [4]). When the values of the exponents for the simple exceptional Lie algebras were announced by Chevalley at the International conference at Cambridge in 1950 Coxeter recognized a rather remarkable coincidence. He observed (1) that in all cases $h\ell = 2r$, so that in our notation $h = s$, and hence $\nu = \omega$ (See Corollary 6.8) (2) $m_i \leq h$ and

(3) the eigenvalues of γ are ω^{m_i}, $i = 1,2,\ldots,\ell$. No proofs were given and the question of obtaining a proof remained open until recently. In [4], using $h\ell = 2r$ as the only empirically observed fact, Coleman proved that $m_i \leq h$ and that the eigenvalues of a Coxeter-Killing transformation are indeed ω^{m_i}, $i = 1,2,\ldots,\ell$. To make the proof independent of any empirical information Coleman states that it would be desireable to prove that $h\ell = 2r$, that is, to prove that the number L of orbits in Δ under the action of γ equals ℓ. (See Corollary 8.2.) Another open question, stated in [4], is proving that $h = q+1$ where q is the order of the highest root. (See [4], p. 356.) But since by definition $s = 1+q$ it is clear that both of these questions are settled when it is shown that $h = s$. The statement that $h = s$ is a part of Theorem 8.4 below.

8.4. Now let γ be defined as in § 8.1. Let $A_\gamma \in G$ be any extension of γ. Let $\Gamma_i \subseteq \Delta$, $i = 1,2,\ldots,L$ be defined as in § 8.2 and let

$$\underline{b}_i = \sum_{\varphi \in \Gamma_i} (e_\varphi). \qquad (8.4.1)$$

By Corollary 8.2 it is clear that each subspace $\underline{b}_i$ is of dimension h. Furthermore since

$$A_\gamma((e_\varphi)) = (e_{\gamma\varphi})$$

for any $\varphi \in \Delta$ it is clear that

$$\underline{g} = \underline{h} + \sum_{i=1}^{L} \underline{b}_i$$

is a direct sum decomposition of $\underline{g}$ into subspaces which are stable under A_γ.

Now since γ^h is the identity element of W we can define the scalar λ_φ, for any root φ, by the relation

$$(A_\gamma)^h e_\varphi = \lambda_\varphi e_\varphi . \qquad (8.4.2)$$

By applying A_γ to both sides of (8.4.2) it is clear that $\lambda\varphi = \lambda_{\gamma\varphi}$ for all φ. Now since A_γ^h reduces to the identity on $\underline{h}$ we can write $A_\gamma^h = \mathrm{Exp}\ x$ for xome $x \in \underline{h}$. It follows that $\lambda_\varphi = e^{(x,\varphi)}$ for any $\varphi \in \Delta$. But then

$$(\lambda_\varphi)^h = \prod_{i=0}^{h-1} \lambda_{\gamma\varphi}$$

$$= e^{(x,\varphi+\gamma\varphi+\cdots+\gamma^{h-1}\varphi)}$$

$$= 1$$

by Lemma 8.2 Thus λ_φ is an h root of unity. This shows that $A_\gamma^{2h} = 1$. We can now prove that h = s and hence that $L = \mathcal{L}$. More than this we have

Theorem 8.4. *Let γ be a Coxeter-Killing transformation on the Cartan subalgebra $\underline{h}$. Let h be the order of γ. Then $h = 1 + \sum_{i=1}^{\ell} q_i$ where the integers q_i are the coefficients of the highest root relative to a basis of simple positive roots. That is, h = s. Moreover, $h\ell = 2r$ where r is the number of positive roots so that there are ℓ distinct orbits $\Gamma_i \subseteq \Delta$, $i = 1,2,\ldots\ell$ in Δ under the action of γ and each orbit contains h roots. Furthermore if we take $\gamma = R_{\alpha_1} R_{\alpha_2} \ldots R_{\alpha_\ell}$ and let φ_i,*

$i = 1,2,\ldots,$ be the positive roots which change sign under γ (see Theorem 8.1) then we can choose an ordering of the orbits so that $\varphi_i \in \Gamma_i$, $i = 1,2,\ldots,\ell$.

Now let $A_\gamma \in G$ be any extension of γ. Then $A_\gamma^h = 1$. That is, $A_\gamma^h e_\varphi = e_\varphi$ for any $\varphi \in \Delta$ so that we can renormalize the root vectors e_φ in such a way that

$$A_\gamma^i e_\varphi = e_{\gamma^i \varphi}$$

for any $\varphi \in \Delta$ and any integer i.

Now let $w_i \in \underline{g}$, $i = 1,2,\ldots,\ell$ be defined by

$$w_i = \sum_{\varphi \in \Gamma_i} e_\varphi$$

and let $\tilde{\underline{h}}$ be the subspace spanned by the elements w_i. Then $\tilde{\underline{h}}$ is a Cartan subalgebra of $\underline{g}$ (so that the elements w_i are semisimple and commute with other). Furthermore $\tilde{\underline{h}} = \underline{g}^{A_\gamma}$.

Proof. Let γ, A_γ and λ_φ for $\varphi \in \Delta$ be as previously defined. It has been shown that $A_\gamma^{2h} = 1$. In particular then A_γ is semisimple. But if $A \in G$ is semisimple it is well known that $\underline{g}^A$ contains a Cartan subalgebra of $\underline{g}$ (See e.g. [8]). For $A = A_\gamma$ the proof is somewhat more direct since A_γ, having finite order, lies in a maximal compact subgroup of G. But then $A_\gamma = \mathrm{Exp}\, x$ for some x lying in a compact form of $\underline{g}$. We could then apply § 6.1(d)). Let $\tilde{\underline{h}}$ be a Cartan subalgebra of $\underline{g}$ contained in $\underline{g}^{A_\gamma}$.

Now let $\underline{b}_i$, $i = 1,2,\ldots, L$ be defined by (8.4.1). Consider the decomposition

$$\underline{g} = \underline{h} + \sum_{i=1}^{L} \underline{b}_i \quad .$$

Obviously

$$\begin{aligned} \underline{g}^{A_\gamma} &= \underline{h} \cap \underline{g}^{A_\gamma} + \sum_{i=1}^{L} \underline{b}_i \cap \underline{g}^{A_\gamma} \\ &= \sum_{i=1}^{L} \underline{b}_i \cap \underline{g}^{A_\gamma} \end{aligned} \tag{8.4.3}$$

by Lemma 8.1.

Now let $\varphi \in \Gamma_i$. As we have already noted $\lambda_\varphi = \lambda_{\gamma^i\varphi}$ for any integer i. Thus A_γ^h reduces to the scalar λ_φ on the space $\underline{b}_i$. But then if $\lambda_\varphi \neq 1$, A_γ^h has no non-zero fixed vectors in $\underline{b}_i$ and hence certainly $\underline{b}_i \cap \underline{g}^{A_\gamma} = 0$. On the other hand if $\lambda_\varphi = 1$ then clearly $\underline{b}_i \cap \underline{g}^{A_\gamma}$ is the one dimensional subspace spanned by $\sum_{i=0}^{h-1} A_\gamma^i e_\varphi$. Thus if L_1 is the number of integers i, $1 \leq i \leq L$ such that $\lambda_\varphi = 1$ for all $\varphi \in \Gamma_i$ it follows from (8.4.3) that $L_1 = \dim \underline{g}^{A_\gamma}$. In fact since $\tilde{\underline{h}} \subseteq \underline{g}^{A_\gamma}$ we then have

$$L \geq L_1 = \dim \underline{g}^{A_\gamma} \geq \ell$$

and hence in particular $L \geq \ell$.

Now we assert that for any i, $\Gamma_i \cap \Delta^+$ and $\Gamma_i \cap \Delta^-$ are both non-empty. Indeed assume, without loss, that $\Gamma_i \cap \Delta^-$ is empty. Then $\sum_{\varphi \in \Gamma_i} \varphi$, in the lexicographical order of h^*, must be strictly positive and hence cannot vanish. This contradicts Lemma 8.1 and hence the assertion is proved. Now since $\Gamma_i \cap \Delta^+$ and $\Gamma_i \cap \Delta^-$ are non-empty it is clear that there exists a root $\varphi \in \Gamma_i \cap \Delta^+$ such that $\gamma\varphi \in \Gamma_i \cap \Delta^-$. That is,

each orbit Γ_i contains at least one positive root which changes sign under γ. But then if we apply Theorem 8.1 it follows obviously that $L \leq \ell$. Thus $L = L_1 = \dim \underline{g}^{A_\gamma} = \ell$. That is, $\lambda_\varphi = 1$ for all $\varphi \in \Delta$ and $\underline{g}^{A_\gamma} = \underline{\tilde{h}}$. Furthermore if the root vectors e_φ, $\varphi \in \Delta$ are normalized so that $A_\gamma e_\varphi = e_{\gamma\varphi}$ it follows that the vectors $w_i = \sum_{\varphi \in \Gamma_i} e_\varphi$ form a basis of $\underline{g}^{A_\gamma}$. Finally observe that since $L = \ell$ each orbit Γ_i contains exactly one positive root which changes sign under γ. Q.E.D.

8.5. Let A_γ be as in Theorem 8.4. As we have noted in §8.3 the eigenvalues of the restriction γ of A_γ to $\underline{h}$ are the numbers ω^{m_i}, $i = 1,2,\ldots,\ell$ where the m_i are the exponents of $\underline{g}$. Let $\underline{m}$ be the B-orthogonal complement to $\underline{h}$ in $\underline{g}$. It is clear that $\underline{m}$ is stable under A_γ. Theorem 8.4 enables us to determine the eigenvalues of A_γ on $\underline{m}$.

Corollary 8.5. *Let A be any extension of the Coxeter-Killing transformation γ and let the root vectors be normalized so that $Ae_\varphi = e_{\gamma\varphi}$ for all $\varphi \in \Delta$. Let $\omega = e^{\frac{2\pi i}{s}}$ and let the roots φ_i be defined as in Theorem 8.1. Define for $i = 1,2,\ldots,\ell$, $j = 1,2,\ldots,s$*

$$y_i^j = \sum_{k=0}^{h-1} \bar{\omega}^{jk} e_{\gamma^k \varphi_i} .$$

Then y_i^j is a basis of the B-orthogonal complement $\underline{m}$ to the Cartan subalgebra $\underline{h}$ in $\underline{g}$ and

$$A_\gamma y_i^j = \omega^j y_i^j$$

for all given values of i and j. In particular the eigenvalues of A_γ on m are ω^j, $j = 0,1,\ldots,s-1$ and each eigenvalue occurs with multiplicity ℓ.

Proof. This is an immediate consequence of Theorem 8.4 which asserts that A_γ on any of the subspaces $\underline{b}_i$ permutes the basal elements $e_{\gamma^k \varphi_i}$, $k = 0,1,2,\ldots,s-1$ according to the cyclic permutation $(1,2,\ldots,s)$. Q.E.D.

8.6. Upon comparing Corollary 8.5 with Theorem 6.8 the suggestion arises that perhaps A_γ is a principle element of $\underline{g}$ and that $\underline{h}$ is in apposition to $\underline{g}^{A_\gamma}$ with respect to A_γ. This is in fact the case as Theorem 8.6 states.

As in Theorem 8.4 write $\tilde{\underline{h}}$ for the Cartan subalgebra $\underline{g}^{A_\gamma}$. Then since all the elements of $\tilde{\underline{h}}$ are fixed by A_γ there exists an element $w \in \tilde{\underline{h}}$ such that

$$A_\gamma = \text{Exp } w \quad .$$

However, such an element w is not unique. We will have to choose w correctly in order to prove Theorem 8.6.

Let $A \in G$ be such that $A\underline{h} = \tilde{\underline{h}}$. Hereafter the addition of the symbol $(\sim)$ to previous notation designates the effect of applying the automorphism A. Now let Y designate the infinite discrete group of translations in $\tilde{\underline{h}}$ by all vectors of the form $\sum_{i=1}^{\ell} t_i \tilde{\epsilon}_i$ where the coefficients t_i are integers and ϵ_i, $i = 1,2,\ldots,\ell$ are defined as in §5.1. It is then easy to prove and well known fact that for x and $y \in \tilde{\underline{h}}$, Exp $2\pi i x =$

$=$ Exp $2\pi i y$ if and only if there exists $\delta \in Y$ such that $\delta x = y$.

For any $\sigma \in \tilde{W}$ and $\delta \in Y$ is obvious that $\sigma\delta\sigma^{-1} \in Y$. It follows that if Z is the set of all linear transformations of $\tilde{\underline{h}}$ of the form $\delta\sigma$, $\delta \in Y$, $\sigma \in \tilde{W}$, then Z is a group (a semi-direct product of Y and $\tilde{W}$). It is well known and not difficult to show that for any $x, y \in \tilde{\underline{h}}$, Exp $2\pi i x$ is conjugate by an element in $N(\tilde{H})$, the normalizer of $\tilde{H}$, to Exp $2\pi i y$ if and only if there exists $\eta \in Z$ such that $\eta x = y$. (See [14], Theorem 6, p. 177).

Consider the action of Z on $\tilde{\underline{h}}^{\#}$. Let

$$T = \{y \in \tilde{\underline{h}} \mid (\tilde{\alpha}_i, y) \geq 0,\ (\tilde{\psi}, y) \leq 1 \text{ for } i = 1,2,\ldots,\ell \} . \tag{8.6.1}$$

The set T is called the fundamental simplex of the chamber $\tilde{D}$. As one knows T has the property that given any $y \in \tilde{\underline{h}}^{\#}$ there exists $x \in T$ and $\eta \in Z$ such that $\eta x = y$. (See [14], Theorem p. 177 and Theorem 8, p. 180). As a consequence if we define

$$U = \bigcup_{\sigma \in W} \sigma T$$

then since $Z = Y\tilde{W}$ given any $y \in \tilde{\underline{h}}^{\#}$, there exists $x \in U$, $\delta \in Y$ such that $\delta x = y$. That is, there exists $x \in U$ such that Exp $2\pi i x =$ Exp $2\pi i y$. Now we can write $A_\gamma =$ Exp $2\pi i y$ for some $y \in \tilde{\underline{h}}$. However, since the eigenvalues of A_γ have modulus one it follows that (y, φ) must be real for every $\varphi \in \tilde{\Delta}$. Thus y must be contained in $\tilde{h}^{\#}$. But then by what we have just

seen we can choose $x \in U$ so that $A_\gamma = \text{Exp } 2\pi i x$. But then $x \in \sigma T$ for some $\sigma \in \tilde{W}$. Now our choice of A was arbitrary except only that $A\underline{h} = \tilde{\underline{h}}$. Without loss of generality then (modification of A) we can assume that $\sigma = 1$. That is, for a suitable choice of A we have

$$A_\gamma = \text{Exp } 2\pi i x$$

for some $x \in T$.

Now by Theorem 8.4 all eigenvalues of A_γ are s roots unity. Thus if we write

$$x = \sum_{i=1}^{\ell} \frac{b_i}{s} \tilde{\epsilon}_i$$

then the scalars b_i are non-negative integers. But now since $(\tilde{\psi}, y) \leq 1$ it follows from (6.3.1) that

$$\sum_{i=1}^{\ell} \frac{q_i b_i}{s} \leq 1 \quad . \tag{8.6.2}$$

But we have more information than this. By Theorem 8.4 A_γ is regular. Thus (φ, y) cannot be an integer for any $\varphi \in \tilde{\Delta}$. Thus the strict inequality holds in (8.6.2). Furthermore $b_i > 0$. Hence we conclude

$$b_i \geq 1$$

and

$$s > \sum_{i=1}^{\ell} q_i b_i \quad .$$

But $s - 1 = q = \sum_{i=1}^{\ell} q_i$ (see (6.3.1)). Hence it follows that $\underline{b_i} = 1$ for all i. That is,

$$x = \sum \frac{\tilde{\epsilon}_i}{s}$$

or

$$x = \frac{\tilde{x}_o}{s} \quad .$$

But $\tilde{x}_o$ is a principle regular element of $\underline{g}$ and

$$A_\gamma = \text{Exp } \frac{2\pi i}{s} \tilde{x}_o \quad .$$

Thus A_γ is a principle element of G. In fact $A_\gamma = APA^{-1}$. Furthermore since the restriction γ of A_γ to $\underline{h}$ has ω for an eigenvalue it follows by definition (See § 7.3) that $\underline{h}$ is in apposition to $\tilde{\underline{h}}$ with respect to A_γ. We have proved

Theorem 8.6. *Let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$. Let W be the Weyl group operating on $\underline{h}$ and let $\gamma \in W$ be a Coxeter-Killing transformation. That is, $\gamma = R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_\ell}$ where α_i, $i = 1,2,\ldots,\ell$ are simple positive roots relative to some lexicographical order in $\underline{h}^*$ and $R_{\alpha_i} \in W$ are the reflections they define.*

Let A_γ be any element of the adjoint group G of $\underline{g}$ which extends γ. Then A_γ is a principle element of G (so that in particular its order is s and all such extensions are conjugate to each other). Let $\tilde{\underline{h}}$ be the set of fixed elements of A_γ. Then $\tilde{\underline{h}}$ is a Cartan subalgebra (since principle elements are regular) and $\underline{h}$ is in apposition to $\tilde{\underline{h}}$ with respect to A_γ (see § 7.3).

We derive a number of corollaries. The first is an immediate consequence of Theorem 7.3 and Theorem 8.6.

Corollary 8.6. *Let h_1' and h_1' be Cartan subalgebras. Assume that h_1' is in apposition to h_1 with respect to the principle element $P \in G$. Then the restriction of P to h_1' defines a Coxeter-Killing transformation of h_1'. In particular the restriction of P_o to h' is a Coxeter-Killing transformation (See Theorem 6.7).*

8.7. In §9.2 we obtain a more general result (Theorem 9.2) than Corollary 8.6.

Taking §2.5(h) into account the next corollary asserts the validity of the empirical method found by A. Shapiro for the determination of the exponents m_i.

Corollary 8.7. *For $i = 1,2,\ldots,\ell$ let k_i be as given as in §6.5 and let m_i be the exponents of g in non-decreasing order. Then $k_i = m_i$, $i = 1,2,\ldots,\ell$.*

In other words if d_i, $i = 1,2,\ldots,\ell$ are the dimensions of the irreducible components of the adjoint representation of a principle TDS on g then

$$\prod_{i=1}^{\ell} (1 + t^{d_i})$$

is the Poincaré polynomial of G.

Proof. This is an immediate consequence of the theorem of Coleman (see § 8.3), Theorem 8.4, Theorem 6.7 and Corollary 8.6. Q.E.D.

The proof of Theorem 8.6 yields a characterization of principle elements of G. We know that a principle element of G

is regular and that its order is s. Among all regular elements in G we now show that its order is minimal.

Corollary 8.6. *Let* $A \in G$ *be regular and let* k *be its order (possibly* ∞*). Then* $k \geq s$ *where* $s = 1 + q$ *and* q *is the order of the highest root. Furthermore* $k = s$ *if and only if* A *is a principle element of* G.

Proof. It suffices to assume the order k of A is finite. Now $\underline{g}^A$ is a Cartan subalgebra. Without loss we may assume this to be $\underline{h}$. Since A has finite order the eigenvalues of A have modulus 1. Thus we may write $A = \text{Exp } 2\pi i x$ where $x \in \underline{h}^{\#}$. More than this, as argued in the proof of Theorem 8.6, by conjugating A, if necessary, we may assume x is contained in the fundamental simplex of the chamber D. That is, $(x,\alpha_i) \geq 0$, $i = 1,2,\ldots,\ell$, $(x,\psi) \leq 1$. On the other hand since A is regular (x,φ) is not an integer for any $\varphi \in \Delta$ so that strict inequalities hold in the inequalities just given. Thus we may write $x = \sum_{i=1}^{\ell} \frac{t_i}{k} \epsilon_i$ where the t_i are positive integers and $\sum_{i=1}^{\ell} t_i q_i < k$. But then

$$s-1 = \sum_{i=1}^{\ell} q_i \leq \sum_{i=1}^{\ell} t_i q_i < k$$

which proves $s \leq k$. Also if $k = s$ then $t_i = 1$ for $i = 1,2,\ldots,\ell$. But in such a case A is clearly principle. In fact $A = P_o$.

9. *A theorem on automorphisms and a characterization of cyclic elements.* 1. Let $S^*(\underline{g})$ be the algebra of all polynomials

on $\underline{g}$ (the symmetric algebra over the dual space to $\underline{g}$). We may write $S^*(\underline{g}) = \sum_{k=0}^{\infty} S^k(\underline{g})$ where $S^k(\underline{g})$ is a space of homogeneous polynomials of degree k. The group G operates on $S^*(\underline{g})$ by the relation

$$A(F)(x) = F(A^{-1}x) \qquad (9.1.1)$$

where $A \in G$, $F \in S^*(\underline{g})$, $x \in \underline{g}$. Let $J^*(\underline{g}) \subseteq S^*(\underline{g})$ be the algebra of polynomials which are left fixed by all $A \in G$ and let $J^k(\underline{g}) = S^k(\underline{g}) \cap J^*(\underline{g})$. Then it is due to Chevalley that $J^*(\underline{g})$ is generated by ℓ algebraically independent homogeneous polynomials I_j, $j = 1,2,\ldots,\ell$. Furthermore if $I_j \in J^{p_j}(\underline{g})$ where $p_1 \leq p_2 \leq \cdots \leq p_\ell$ then $p_i = m_i + 1$. (See [3] and also [2].)

The following lemma is an immediate consequence of this result of Chevalley and Corollary 8.7.

Lemma 9.1. *Let I_j, $j = 1,2,\ldots,\ell$ be the invariant polynomials defined as above. Then* deg $I_\ell = s$ *and* deg $I_j < s$ *for all* $j < \ell$.

Now let $J_+^*(\underline{g})$ be the hyperplane in $J^*(\underline{g})$ consisting of all polynomials in $J(\underline{g})$ with zero constant term. Consider the variety in $\underline{g}$ formed by the zeros of all polynomials in $J_+^*(\underline{g})$. As one might expect,

Theorem 9.1. *Let $x \in \underline{g}$. Then $F(x) = 0$ for all $F \in J_+^*(\underline{g})$ if and only if x is nilpotent.*

Proof. Let $F_k \in S^k(\underline{g})$ be defined by

$$F_k(y) = \text{trace } (\text{ad } y)^k \quad .$$

It is clear that $F_k \in J^k(\underline{g})$. It is also clear that if x is not nilpotent $F_k(x) \neq 0$ for some integer k. Hence to prove Theorem 9.1 it must be shown, conversely, that if x is nilpotent $F(x) = 0$ for all $F \in J^*(\underline{g})$. Assume then that x is nilpotent. By Lemma 3.3 there exists $y \in \underline{g}$ such that $[y,x] = x$. Let λ be an arbitrary non-zero real scalar and let $A^\lambda = \text{Exp } \lambda y$. Clearly $A^\lambda x = e^\lambda x$. Now let $F \in J^k(\underline{g})$. Obviously by (9.1.1)

$$\begin{aligned} F(x) &= F(A^\lambda x) \\ &= F(e^\lambda x) \\ &= e^{k\lambda} F(x) \quad . \end{aligned}$$

But $e^{k\lambda} \neq 1$. Thus $F(x) = 0$. This proves Theorem 9.1 since $J(\underline{g})$ is generated by homogeneous polynomials. Q.E.D.

9.2. In the proof of Theorem 9.1 we have seen that if x is nilpotent and c is any non-zero scalar there exists an automorphism $A \in G$ such that $Ax = cx$. The situation as Theorem 9.2 indicates is entirely different when x is not nilpotent. If x is cyclic it has been shown (see Theorem 6.7) that there exists a principle element $P \in G$ such that $Px = \mu x$ where μ is a primitive s root of unity. Among all possible relations of the form $Ax = cx$ where $A \in G$, and x is not nilpotent it will be shown that this is, in a very definite way, the critical case.

Theorem 9.2. Assume $A \in G$, $x \in \underline{g}$ and $Ax = cx$. Assume further that x is not nilpotent. Then c is a primitive m^{th} root

of unity for some integer m. *Furthermore* m *divides* m_j+1 *for some exponent* m_j *of* g *so that in particular* $m \leq s$. *Finally if* $m = s$ *then* (1) x *is a cyclic element of* g *and* (2) A *is a principle element of* G. *Moreover if* h_1 *is the Cartan subalgebra which contains* x (*cyclic implies regular*) *then* h_1 *is stable under* A *and the restriction* σ *of* A *to* h_1 *is a Coxeter-Killing transformation.*

Proof. Since x is not nilpotent it follows from Theorem 9.1 that there exists j, $1 \leq j \leq \ell$, such that $I_j(x) \neq 0$. But

$$\begin{aligned} I_j(x) &= I_j(Ax) \\ &= I_j(cx) \\ &= c^{p_j} I_j(x) \quad . \end{aligned}$$

Thus $c^{p_j} = 1$. This proves that c is a primitive m^{th} root of unity where m is an integer which divides $p_j = m_j+1$. By Lemma 9.1, $m \leq s$.

Now let $S^*(\underline{h})$ be the algebra of polynomials on $\underline{h}$ and let $J^*(\underline{h})$ be the subalgebra of all polynomials which are fixed under the action of the Weyl group W. Let $\hat{\chi} : S^*(\underline{g}) \longrightarrow S^*(\underline{h})$ be the homomorphism defined by restricting a polynomial on $\underline{g}$ to $\underline{h}$ then it is a known result of Chevalley that the restriction

$$\chi : J^*(\underline{g}) \longrightarrow J^*(\underline{h}) \qquad (9.2.1)$$

of $\hat{\chi}$ to $J^*(\underline{g})$ defines an isomorphism of $J^*(\underline{g})$ onto $J^*(\underline{h})$. (See [3], § IV; also [12], Theorème 2, p. 19-10).

To prove the remaining part of Theorem 9.2 we shall need

Lemma 9.2. *Let* $I_j \in J^*(g)$, $j = 1,2,\ldots,\ell$ *be given as in* § 9.1. *Let* $x,y \in \underline{h}$. *Then* x *and* y *are conjugate with respect to* W *if and only if* $I_j(x) = I_j(y)$ *for* $j = 1,2,\ldots,\ell$.

Proof. It is obvious from the definition of W that if x and y are conjugate with respect to W, $I_j(x) = I_j(y)$ for $j = 1,2,\ldots,\ell$. Conversely, assume $I_j(x) = I_j(y)$, $j=1,2,\ldots,\ell$. Let $x_1, x_2, \ldots, x_{r_1}$ and $y_1, y_2, \ldots, y_{r_2}$ be, respectively, the distinct conjugates of x and y under W. Assume x is not conjugate to y. Then x_i, y_j, $i = 1,2,\ldots,r_1$, $j = 1,2,\ldots,r_2$ represent $r_1 + r_2$ points in $\underline{h}$. Since points in $\underline{h}$ are obviously algebraic varieties there exists $F \in S^*(\underline{h})$ such that $F(x_i) = 0$, $i = 1,2,\ldots,r_1$, $F(y_j) = 0$, $j = 2,\ldots,r_2$ and $F(y_1) = 1$. But then if

$$\tilde{F} = \sum_{\sigma \in W} \sigma(F)$$

it follows that $\tilde{F} \in J^*(\underline{h})$, $\tilde{F}(y) \neq 0$ whereas $\tilde{F}(x) = 0$. But $\chi(I_j)$, $j = 1,2,\ldots,\ell$ generate $J^*(\underline{h})$ (see (9.2.1). Thus there exists j such that $I_j(x) \neq J_j(y)$. This is a contradiction.

Q.E.D.

Proof of Theorem 9.2 *continued*. Assume m = s. Then $c^{p_j} = 1$ implies $j = \ell$ by Lemma 9.1. Thus we conclude $I_j(x) = 0$ for $j = 1,2,\ldots,\ell-1$ and $I_\ell(x) \neq 0$.

Now it is well known (e.g. see [9], Theorem 6) that we may write $x = x_1 + x_2$, uniquely, where x_1 is semi-simple, x_2 is nilpotent and $[x_1,x_2] = 0$. By the uniqueness of this decomposition it is clear that $Ax_1 = cx_1$ and $Ax_2 = cx_2$. Furthermore by

hypothesis we know that $x_1 \neq 0$. Thus if we apply the previous argument to the relation $Ax_1 = cx_1$ it follows that $I_j(x_1) = 0$ for $j = 1,2,\ldots,\ell-1$, and $I_\ell(x_1) \neq 0$.

Now since x_1 is semi-simple it lies in a Cartan subalgebra. Without loss of generality we may assume $x_1 \in \underline{h}$. Now let γ be the Coxeter-Killing transformation given as in §8.1. By Corollary 8.6, Theorem 7.3 and Theorem 6.7 there exists a cyclic vector $z \in \underline{h}$ such that $\gamma z = \omega z$. If we apply the argument above once more to the relation $A_\gamma z = \omega z$, where A_γ is any extension of γ, it follows that $I_j(z) = 0$ for $j = 1,2,\ldots,\ell-1$ and $I_\ell(z) \neq 0$. But then by Lemma 9.2, x_1 must be conjugate to a scalar multiple of z. Thus x_1 is cyclic. But then x_1 is regular. Since $[x_1,x_2] = 0$ this implies $x_2 \in \underline{h}$. But x_2 is nilpotent. Hence $x_2 = 0$ or $x = x_1$. Thus x is cyclic.

Now since the regular vector $x \in \underline{h}$ is an eigenvector of A it follows that $\underline{h}$ is stable under A. Let $\tau \in W$ be the restriction of A to $\underline{h}$. By Lemma 7.1 ω is an eigenvalue of τ. Let $y \in \underline{h}$ be the corresponding eigenvector. The argument above applied to y proves that y is cyclic and when properly normalized there exists $\sigma \in W$ such that $\sigma z = y$. Now let $\gamma_1 = \sigma\gamma\sigma^{-1}$. Then γ_1 is a Coxeter-Killing transformation and $\gamma_1 y = \omega y$. But then $\tau^{-1}\gamma_1$ leaves y fixed. But it is well known that the only element of W which leaves a regular element fixed is the identity (see [8]). Thus τ is a Coxeter-Killing transformation and since A is an extension of τ it follows from Theorem 8.6 that A is a principle element of G. Q.E.D.

Observe that Theorem 9.2 is a generalization of Corollary 8.6.

In [4] an important role is played by a regular eigenvector of γ. It is easy to see (see [4]) that the eigenvalue corresponding to any regular eigenvector of γ is a primitive s root of unity. By Theorem 9.2 we see that all such eigenvectors are cyclic.

Corollary 9.2 characterizes a Coxeter-Killing transformation among all elements in W.

Corollary 9.2. *Let $\sigma \in W$ be arbitrary and let c be an eigenvalue of σ. Then c is a primitive m^{th} root of unity where m is an integer which divides $m_j + 1$ for some exponent m_j of g. In particular $m \leq s$. Moreover, if $m = s$ then σ is a Coxeter-Killing transformation and the eigenvector of σ corresponding to c is a cyclic element of g.*

9.3. A corollary of the proof of Theorem 9.2 yields a characterization of cyclic elements in g.

Corollary 9.3. *Let I_j, $j = 1,2,\ldots,\ell$ be the invariant polynomials on g given as in §9.1. Let $x \in g$. Then x is cyclic if and only if*

$$I_j(x) = 0 \qquad \text{for } j = 1,2,\ldots,\ell-1$$

and

$$I_\ell(x) \neq 0 \quad .$$

Proof. The proof that the equations above hold when x is cyclic has been given in the proof of Theorem 9.2. Assume,

conversely, that these equations hold. Write

$$x = x_1 + x_2 \tag{9.3.1}$$

where x_1 is semi-simple, x_2 is nilpotent and $[x_1,x_2] = 0$. (See [9], Theorem 6), By Theorem 9.1 $x_1 \neq 0$. Without loss of generality we may assume $x_1 \in \underline{h}$. Consider $\underline{g}^{x_1}$. It is clear that $\underline{g}^{x_1}$ is generated linearly by $\underline{h}$ and all root vectors e_φ where $(\varphi,x_1) = 0$. It follows easily then that $\underline{g}^{x_1} = \underline{c} + [\underline{g}^{x_1},\underline{g}^{x_1}]$ is a direct sum (see [9], Theorem 7) where $\underline{c}$ is the center of $\underline{g}^{x_1}$. Furthermore $[\underline{g}^{x_1},\underline{g}^{x_1}]$ is semi-simple and $\underline{c} \subsetneq \underline{h}$. Now $x_2 \in \underline{g}^{x_1}$. Write $x_2 = y_1 + y_2$ where $y_1 \in \underline{c}$ and $y_2 \in [\underline{g}^{x_1},\underline{g}^{x_1}]$. It follows immediately that y_2 is a nilpotent element of $[\underline{g}^{x_1},\underline{g}^{x_1}]$ and hence y_2 is a nilpotent element of $\underline{g}$. (See footnote 1.)

But $x_1 + y_1 \in \underline{h}$ is semi-simple. Writing $x = (x_1 + y_1) + y_2$ gives a second decomposition of the form (9.3.1). By uniqueness $y_1 = 0$ and hence $x_2 \in [\underline{g}^{x_1},\underline{g}^{x_1}]$. Applying Lemma 3.3 there exists $y \in [\underline{g}^{x_1},\underline{g}^{x_1}]$ such that $[y,x_2] = x_2$. Of course $[y,x_1] = 0$. Thus if $A^\lambda = \operatorname{Exp} \lambda y$ it is clear that

$$A^\lambda x = x_1 + e^\lambda x_2 \quad .$$

But for any $F \in J^*(\underline{g})$, $F(x) = F(A^\lambda x)$. By choosing λ such that e^λ is arbitrarily close to 0 it follows from continuity that $F(x) = F(x_1)$. In particular $I_j(x_1) = 0$ for $j = 1,2,\ldots,\ell-1$ and $I_\ell(x_1) \neq 0$. But then as in the proof of Theorem 9.2 it follows that x_1 is cyclic. But then x_1 is regular and hence $[x_1,x_2] = 0$ implies $x_2 \in \underline{h}$. Since x_2 is nilpotent this implies $x_2 = 0$ and

hence $x = x_1$ is cyclic. Q.E.D.

Even without knowledge of Corollary 9.2 it is interesting to observe that the condition of that corollary, which singles out I_{ℓ}, is independent of the set of generators I_j of $J^*(\underline{g})$ so long as they are homogeneous, algebraically independent and ordered the way they are. This would not be necessarily true if I_j was singled out where $j < \ell$.

REFERENCES

1. A. Borel, Groupes linéaires algébriques, Ann. of Math., vol. 64 (1956), pp. 20-82.

2. H. Cartan, La transgression dans un group de Lie et dans un espace fibre principle, Colloque de Topologie, Bruxelles, 1950, pp. 57-71.

3. C. Chevalley, The Betti numbers of the exceptional simple Lie groups, Proc. Inter. Cong. Math., Cambridge, vol. II (1950), pp. 21-24.

4. A. J. Coleman, The Betti numbers of the simple Lie groups, Can. J. Math., vol. 10 (1958), pp. 349-356.

5. H. S. M. Coxeter, The product of the generators of a finite group generated by reflections, Duke Math. J., vol. 18 (1951), pp. 765-782.

6. E. B. Dynkin, Some properties of the system of weights of a linear transformation of a semisimple Lie group, Dokl. Akad. Nauk SSSR (N.S.) 71 (1950), pp. 221-224 (Russian).

7. E. B. Dynkin, Semisimple subalgebras of semisimple Lie algebras, Amer. Math. Soc. Translations, vol. 6 (1957), pp. 111-244.

8. F. Gantmacher, Canonical representation of automorphisms of a complex semi-simple Lie group, Matematiceski Sbornik, vol. 47 (1939), pp. 101-146.

9. N. Jacobson, Completely reducible Lie algebras of linear transformations, Proc. Amer. Math. Soc., vol. 2 (1951), pp. 105-113.

10. A. I. Mal'cev, On semi-simple subgroups of Lie groups, Amer. Math. Soc. Translations, no. 33 (1950).

11. V. V. Morosov, On a nilpotent element of a semi-simple Lie algebra, C. R. (Doklady) Acad. Sci. URSS. vol. 36 (1942), pp. 83-86.

12. Séminaire Sophus Lie, le année 1954/1955, Théorie des algèbre de Lie, Topologie des groupes de Lie, Ecole Normale Supérieure, Paris, 1955.

13. J. de Siebenthal, Sur certaines sous-groupes de rang un des groupes de Lie clos, C. R. Acad. Sci. Paris 230 (1950), pp. 910-912.

14. H. Weyl, The structure and representation of continuous groups, Inst. for Adv. Study notes by R. Brauer, 1934-1935, reprinted 1955.